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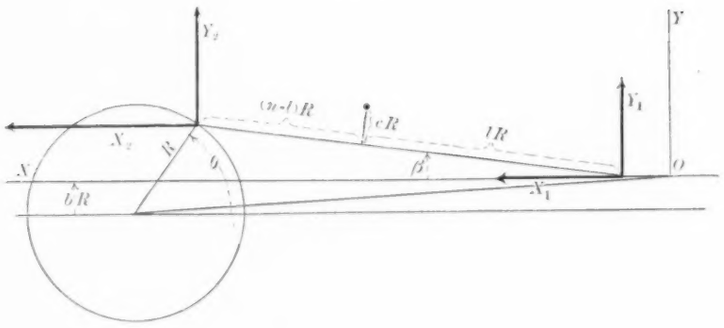
FIG. 1.



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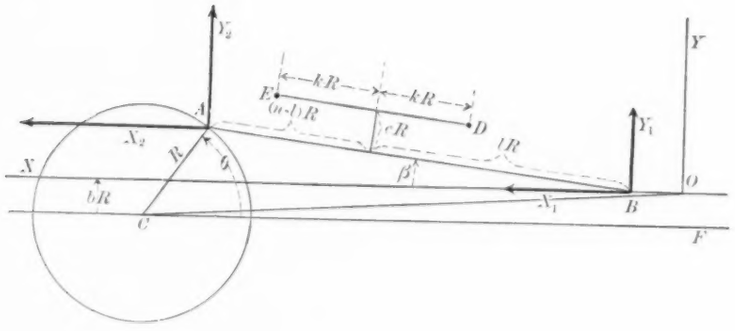

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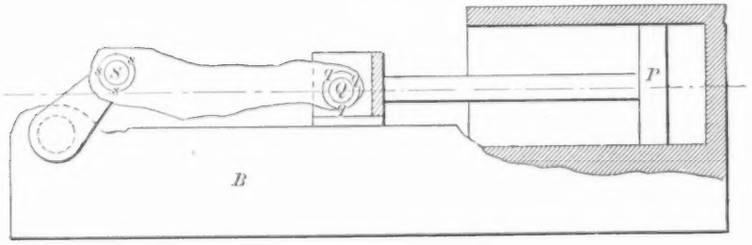

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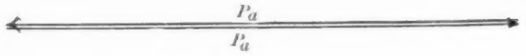

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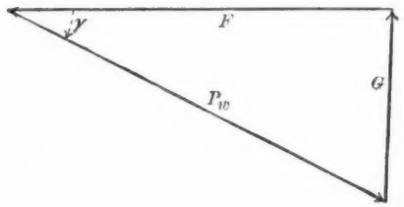

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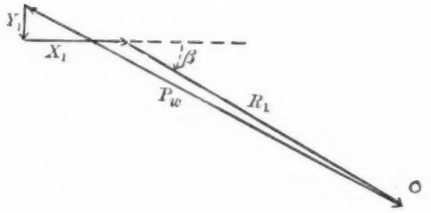

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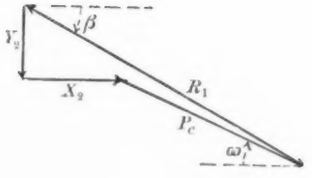

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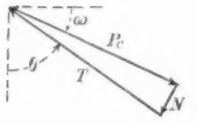

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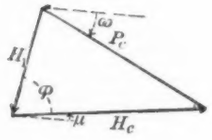

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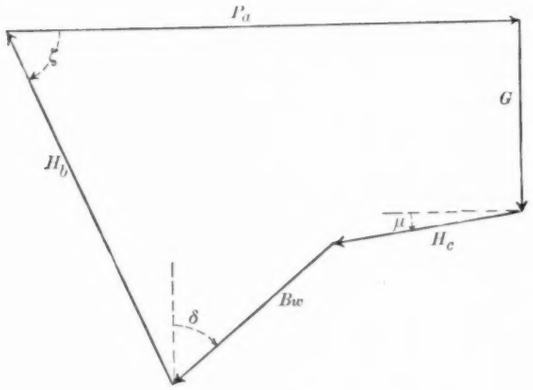
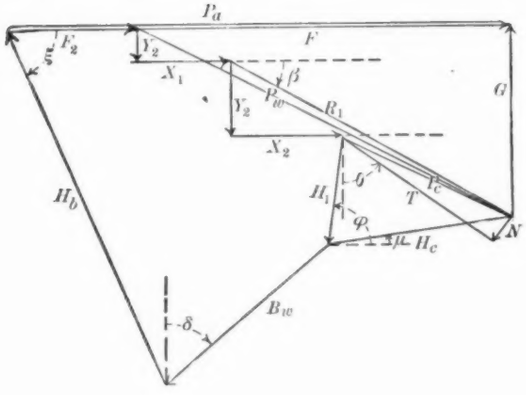


FIG. 12.



ANNALS OF MATHEMATICS.

VOL. V. FEBRUARY, 1890. NO. 3.

GENERAL SOLUTION OF THE TRANSMISSION OF FORCE IN A STEAM ENGINE, INCLUDING FRICTION, ACCELERATION, AND GRAVITY.

By PROF. D. S. JACOBUS,* Hoboken, N. J.

This problem has been discussed by many, but the author has met with no perfectly general solution. Various approximate ones are employed in general engineering work involving errors of unknown magnitude. The author has, therefore, endeavored in this paper to present a general set of equations by means of which the errors of the more approximate ones may be ascertained.

The following general conditions have been assumed:

(*a*) that the centre of the crank shaft is not necessarily on the line of travel of the wrist-pin;†

(*b*) that the centre of gravity of the connecting-rod is not necessarily in its line of centres;

(*c*) that the crank revolves at a uniform speed;

(*d*) that the mass of the moving parts is distributed in any manner;

(*e*) that the line of travel of the wrist pin is not necessarily horizontal or vertical;

(*f*) that there is friction between the parts.

Fig. 1 represents the main lines of an engine referred to the horizontal axis OX, coinciding with the line of travel of the wrist-pin, O being the extreme point of its travel and OY the vertical axis; the directions of the various quantities are indicated by arrow-heads.

NOTATION.

Let F_1 = the force to accelerate the piston, piston-rod, and cross-head;

* I am greatly indebted to Prof. J. Burkitt Webb for valuable assistance in preparing this article, many of the equations having been worked out independently by him, and compared in order to check the final result. The methods employed in the solutions have also been in a great measure suggested by him.

† Engines are now in use that require this condition to enter the equations: for example, the Westinghouse.

X_1, X_2 and Y_1, Y_2 = the x and y components of the forces which accelerate the connecting-rod, applied at the wrist and crank-pins respectively;

R = radius of the crank;

nR = length of the line joining the wrist and crank-pin centres;

lR = distance from the wrist-pin to the foot of the perpendicular let fall from the centre of gravity of the connecting-rod upon this line;

cR = length of this perpendicular;

bR = perpendicular from the centre of the crank shaft upon the line of motion of the wrist-pin;

θ = crank angle;

β = connecting-rod angle;

M = mass of piston, piston-rod, and cross-head;

m = mass of the connecting-rod, and

kR = its principal radius of gyration;

W_1 = weight of the piston, piston-rod, and cross-head;

W_2 = weight of the connecting-rod;

τ = angular velocity of crank;

s = length of stroke;

t = time required by the crank to turn through the angle θ.

FORCE REQUIRED TO ACCELERATE THE PISTON, PISTON-ROD, AND CROSS-HEAD.

The general value of x, the co-ordinate of the wrist-pin, is

$$x = \sqrt{(R + nR)^2 - b^2R^2} - R(\cos\theta + n\cos\beta). \tag{1}$$

From the figure we have

$$\sin\beta = \frac{\sin\theta - b}{n}; \tag{2}$$

therefore,

$$\cos\beta = \frac{1}{n}\sqrt{n^2 - (\sin\theta - b)^2}, \tag{3}$$

which reduces (1) to

$$x = \sqrt{(R + nR)^2 - b^2R^2} - R[\cos\theta + \sqrt{n^2 - (\sin\theta - b)^2}]. \tag{4}$$

Differentiating, we have

$$\frac{dx}{d\theta} = R\left(\sin\theta + \frac{(\sin\theta - b)\cos\theta}{\sqrt{n^2 - (\sin\theta - b)^2}}\right),$$

and

$$\frac{d^2x}{d\theta^2} = R\left(\cos\theta - \frac{\sin\theta(\sin\theta - b)}{\sqrt{n^2 - (\sin\theta - b)^2}} + \frac{n^2\cos^2\theta}{[n^2 - (\sin\theta - b)^2]^{\frac{3}{2}}}\right).$$

For brevity, let

$$\frac{n^2 \cos^2\theta}{[n^2 - (\sin\theta - b)^2]^{\frac{3}{2}}} - \frac{\sin\theta\,(\sin\theta - b)}{\sqrt{n^2 - (\sin\theta - b)^2}} = Z; \tag{5}$$

then

$$\frac{d^2x}{d\theta^2} = R\,(\cos\theta + Z). \tag{6}$$

But

$$\tau = \frac{d\theta}{dt}, \tag{7}$$

by means of which $d\theta$ may be eliminated from (6), giving

$$\frac{d^2x}{dt^2} = \tau^2 R\,(\cos\theta + Z);$$

from which it follows that

$$F_1 = M\tau^2 R\,(\cos\theta + Z). \tag{8}$$

FORCES REQUIRED TO ACCELERATE THE CONNECTING-ROD.

The general equations for the forces to accelerate the connecting-rod have been derived by two methods:

1. By determining the forces at the centre of gravity required to vary the translation of the mass, and the moment necessary for its angular acceleration;

2. By assuming the mass of the rod to be symmetrically arranged, say in two equal and opposite portions, at a distance from the centre of gravity equal to the radius of gyration, and determining the forces and moments required for the acceleration of such a representative rod.

FIRST METHOD.

The motion of the connecting-rod is composed of a translation of its centre of gravity and a rotation of the rod about it, this translation and rotation being so related as to cause the wrist pin to follow a right line, and the crank pin a circle. This motion might be considered as a rectilinear translation of the wrist pin and a rotation about it, but is not so taken, because the centrifugal forces of the parts of the rod are not balanced about that point and complicate the problem.

To determine the forces at the wrist and crank pins required to accelerate

the centre of gravity, we determine the force necessary when applied at that point, and resolve it into two equivalent forces applied at the wrist and crank pins, which latter forces will be inversely proportional to their lever arms about the centre of gravity. We thus obtain forces at the pins that will produce translation only.

To determine the rotative forces we find the moment necessary to produce the rotation, and then place such equal and opposite forces at the pins as will produce it.

The sum of the translative and rotative forces at the pins will be the total forces required to produce the complete acceleration of the rod.

Let x and y be the co-ordinates of the centre of gravity; (see Fig. 1) then we have

$$x = \sqrt{(R + nR)^2 - b^2 R^2} - R[\cos\theta + (n - l)\cos\beta + c\sin\beta].$$

Substituting the values of $\sin\beta$ and $\cos\beta$ from (2) and (3), we obtain

$$x = \sqrt{(R + nR)^2 - b^2 R^2} - R\left(\cos\theta + \frac{n - l}{n}\sqrt{n^2 - (\sin\beta - b)^2} + \frac{c(\sin\theta - b)}{n}\right);$$

from which

$$\frac{dx}{d\theta} = R\left(\sin\theta + \frac{(n - l)(\sin\theta - b)\cos\theta}{n\sqrt{n^2 - (\sin\theta - b)^2}} - \frac{c\cos\theta}{n}\right),$$

and

$$\frac{d^2x}{d\theta^2} = R\left(\cos\theta - \frac{(n - l)(\sin\theta - b)\sin\theta}{n\sqrt{n^2 - (\sin\theta - b)^2}} + \frac{(n - l)n\cos\theta}{[n^2 - (\sin\theta - b)^2]^{\frac{3}{2}}} + \frac{c\sin\theta}{n}\right).$$

Introducing the values of Z and t from (5) and (7), we have

$$\frac{d^2x}{dt^2} = \tau^2 R\left(\cos\theta + \frac{n - l}{n}Z + \frac{c}{n}\sin\theta\right).$$

Let F_x be the x component of the translative force at the centre of gravity; then

$$F_x = m\tau^2 R\left(\cos\theta + \frac{n - l}{n}Z + \frac{c}{n}\sin\theta\right). \qquad (9)$$

From Fig. 1 we have for the ordinate of the centre of gravity

$$y = lR\sin\beta + cR\cos\beta = \frac{lR}{n}(\sin\theta - b) + \frac{cR}{n}\sqrt{n^2 - (\sin\theta - b)^2}.$$

Differentiating, we have

$$\frac{dy}{d\theta} = R\left(\frac{l}{n}\cos\theta - \frac{c(\sin\theta - b)\cos\theta}{n\sqrt{n^2-(\sin\theta-b)^2}}\right),$$

and

$$\frac{d^2y}{d\theta^2} = -R\left(\frac{l}{n}\sin\theta - \frac{c(\sin\theta-b)\sin\theta}{n\sqrt{n^2-(\sin\theta-b)^2}} + \frac{cn\cos^2\theta}{[n^2-(\sin\theta-b)^2]^{\frac{3}{2}}}\right);$$

from which

$$\frac{d^2y}{dt^2} = -\tau^2 R\left(\frac{l}{n}\sin\theta + \frac{c}{n}Z\right).$$

Let F_y be the Y component of the translative force at the centre of gravity; then

$$F_y = -m\tau^2 R\left(\frac{l}{n}\sin\theta + \frac{c}{n}Z\right). \tag{10}$$

In order to determine the rotative forces, we proceed as follows:
From (2) we have

$$\frac{d\beta}{d\theta} = \frac{\cos\theta}{n\cos\beta},$$

$$\frac{d^2\beta}{d\theta^2} = -\frac{\sin\theta}{\sqrt{n^2-(\sin\theta-b)^2}} + \frac{\cos^2\theta(\sin\theta-b)}{[n^2-(\sin\theta-b)^2]^{\frac{3}{2}}};$$

from which it follows that the moment required to produce the rotation will be

$$M_r = -m\tau^2 R^2 k^2\left(\frac{\sin\theta}{\sqrt{n^2-(\sin\theta-b)^2}} - \frac{\cos^2\theta(\sin\theta-b)}{[n^2-(\sin\theta-b)^2]^{\frac{3}{2}}}\right). \tag{11}$$

Let D_w, D_c and E_w, E_c be the horizontal and vertical forces acting at the wrist and crank pins, respectively, into which the forces F_x and F_y are divided; then we have

$$D_w nR\sin\beta = F_x[(n-l)R\sin\beta - cR\cos\beta],$$

and

$$D_c nR\sin\beta = F_x(lR\sin\beta + cR\cos\beta);$$

from which

$$D_w = F_x\left(\frac{n-l}{n} - \frac{c}{n}\cot\beta\right) = F_x\left(\frac{n-l}{n} - \frac{c\sqrt{n^2-(\sin\theta-b)^2}}{n(\sin\theta-b)}\right), \tag{12}$$

and

$$D_c = F_x\left(\frac{l}{n} + \frac{c}{n}\cot\beta\right) = F_x\left(\frac{l}{n} + \frac{c\sqrt{n^2-(\sin\theta-b)^2}}{n(\sin\theta-b)}\right). \tag{13}$$

Dividing the vertical force F_y between the wrist and crank pins in the same way, we have

$$F_y\,[(n-l)\,R\cos\beta + cR\sin\beta] = E_w n R\cos\beta,$$

and

$$F_y\,(lR\cos\beta - cR\sin\beta) = E_c n R\cos\beta;$$

from which

$$E_w = F_y\left\{\frac{n-l}{n} + \frac{c\,(\sin\theta - b)}{n\sqrt{n^2-(\sin\theta-b)^2}}\right\}, \tag{14}$$

and

$$E_c = F_y\left\{\frac{l}{n} - \frac{c\,(\sin\theta - b)}{n\sqrt{n^2-(\sin\theta-b)^2}}\right\}. \tag{15}$$

The value of M_r given in (11) may be written

$$M_r = mr^2R^2\frac{k^2}{n^2}Zn\sin\beta - mr^2R^2\frac{k^2}{n^2}\sin\theta\,.\,n\cos\beta;$$

therefore the rotation may be produced by two equal and opposite forces in the X direction at the wrist and crank pins equal to $\pm mr^2R\frac{k^2}{n^2}Z$, together with a second set applied in the y direction at the same points equal to $\pm mr^2R\frac{k^2}{n^2}\sin\theta$.*

Adding these forces to those required to produce the translation, we have the total accelerating forces,

$$Y_1 = E_w + mr^2R\frac{k^2}{n^2}\sin\theta, \tag{16}$$

$$Y_2 = E_c - mr^2R\frac{k^2}{n^2}\sin\theta, \tag{17}$$

$$X_1 = D_w + mr^2R\frac{k^2}{n^2}Z, \tag{18}$$

$$X_2 = D_c - mr^2R\frac{k^2}{n^2}Z. \tag{19}$$

* This division of M_r reduces the values of the accelerating forces to a form in which the value of Z may be employed and numerical computations thus facilitated.

Substituting the values of E_w, E_c, D_w and D_c as given in equations (12)–(15), in equations (16)–(19) the accelerating forces become

$$Y_1 = -mr^2R\left[\left(\frac{n-l}{n}+\frac{c(\sin\theta-b)}{n\sqrt{n^2-(\sin\theta-b)^2}}\right)\left(\frac{l}{n}\sin\theta+\frac{c}{n}Z\right)-\frac{k^2}{n^2}\sin\theta\right], \quad (20)$$

$$Y_2 = -mr^2R\left[\left(\frac{l}{n}-\frac{c(\sin\theta-b)}{n\sqrt{n^2-(\sin\theta-b)^2}}\right)\left(\frac{l}{n}\sin\theta+\frac{c}{n}Z\right)+\frac{k^2}{n^2}\sin\theta\right], \quad (21)$$

$$X_1 = mr^2R\left[\left(\frac{n-l}{n}-\frac{c\sqrt{n^2-(\sin\theta-b)^2}}{n(\sin\theta-b)}\right)\left(\cos\theta+\frac{n-l}{n}Z+\frac{c}{n}\sin\theta\right)+\frac{k^2}{n^2}Z\right], \quad (22)$$

$$X_2 = mr^2R\left[\left(\frac{l}{n}+\frac{c\sqrt{n^2-(\sin\theta-b)^2}}{n(\sin\theta-b)}\right)\left(\cos\theta+\frac{n-l}{n}Z+\frac{c}{n}\sin\theta\right)-\frac{k^2}{n^2}Z\right]. \quad (23)$$

SECOND METHOD.

Fig. 2 represents the main lines of an engine in accordance with the second supposition; kR, kR are two radii of gyration drawn from the centre of gravity parallel to the line AB joining the centres of the crank and wrist pins, and the mass is supposed to be concentrated in two equal portions at their extremities D and E. This reduces the consideration of the motion of the rod to the more simple discussion of that of the two concentrated masses.

Let the co-ordinates of the masses D and E be x_1, y_1 and x_2, y_2, respectively then we have

$$x_1 = C_1 - R\left[\cos\theta + (n-l+k)\cos{}_1\beta + c\sin{}_1\beta\right],$$

in which C_1 is the distance from the centre O to the ordinate of D when the crank is on the inner dead centre.

Differentiating the above value of x_1, we have

$$\frac{d^2x_1}{d\theta^2} = R\left[\cos\theta + (n-l+k)\left(\cos{}_1\beta\frac{d_1\beta^2}{d\theta^2}+\sin{}_1\beta\frac{d^2{}_1\beta}{d\theta^2}\right) + c\left(\sin{}_1\beta\frac{d_1\beta^2}{d\theta^2}-\cos{}_1\beta\frac{d^2{}_1\beta}{d\theta^2}\right)\right]. \quad (24)$$

It also follows from Fig. 2 that

$$\sin\theta - b = n\sin {}_1\beta,$$

from which

$$\frac{d_1\beta}{d\theta} = \frac{\cos\theta}{\sqrt{n^2-(\sin\theta-b)^2}}, \tag{25}$$

and

$$\frac{d^2{}_1\beta}{d\theta^2} = \frac{\cos^2\theta\,(\sin\theta-b)}{[n^2-(\sin\theta-b)^2]^{\frac{3}{2}}} - \frac{\sin\theta}{\sqrt{n^2-(\sin\theta-b)^2}}. \tag{26}$$

Substituting the values of $\frac{d_1\beta}{d\theta}$ and $\frac{d^2{}_1\beta}{d\theta^2}$ in (24), we have

$$\frac{d^2x_1}{d\theta^2} = R\left[\cos\theta + \frac{n-l+k}{n}\left(\frac{n^2\cos^2\theta}{[n^2-(\sin\theta-b)^2]^{\frac{3}{2}}} - \frac{\sin\theta\,(\sin\theta-b)}{\sqrt{n^2-(\sin\theta-b)^2}}\right) + \frac{c\sin\theta}{n}\right],$$

and on introducing the value of Z, given in (5), this becomes

$$\frac{d^2x_1}{d\theta^2} = R\left(\cos\theta + \frac{n-l+k}{n}Z + \frac{c\sin\theta}{n}\right).$$

Let F_{dx} be the force required to produce this acceleration, and we have

$$F_{dx} = \frac{1}{2}mr^2R\left(\cos\theta + \frac{n-l+k}{n}Z + \frac{c\sin\theta}{n}\right). \tag{27}$$

Dividing this between the crank and wrist pins gives for the portions acting at the crank pin A

$$F_{dx}A = \frac{1}{2}mr^2R\left[\frac{l-k+c\cot {}_1\beta}{n}\left(\cos\theta + \frac{n-l+k}{n}Z + \frac{c\sin\theta}{n}\right)\right]. \tag{28}$$

The force required to produce the X acceleration of the mass E, obtained in a similar way to F_{dx}, is

$$F_{ex} = \frac{1}{2}mr^2R\left(\cos\theta + \frac{n-l-k}{n}Z + \frac{c\sin\theta}{n}\right), \tag{29}$$

and its component acting at the crank pin will be

$$F_{ex}A = \frac{1}{2}mr^2R\left[\frac{l+k+c\cot {}_1\beta}{n}\left(\cos\theta + \frac{n-l-k}{n}Z + \frac{c\sin\theta}{n}\right)\right]. \tag{30}$$

Adding these two components as given in equations (28) and (30), we have the total accelerating force X_2 at the crank pin, or

$$F_{dx}A + F_{ex}A = X_2;$$

from which

$$X_2 = m\tau^2 R\left[\left(\frac{l}{n} + \frac{c\sqrt{n^2-(\sin\theta-b)^2}}{n(\sin\theta-b)}\right)\left(\cos\theta + \frac{n-l}{n}Z + \frac{c\sin\theta}{n}\right) - \frac{k^2}{n^2}Z\right]. \quad (31)$$

The accelerating force X_1 acting at the wrist pin may be found by obtaining the sum of the forces in the X direction at the masses D and E and subtracting X_2, or

$$X_1 = F_{dx} + F_{ex} - X_2;$$

from which

$$\begin{aligned} X_1 &= m\tau^2 R\left(\cos\theta + \frac{n-l}{n}Z + \frac{c}{n}\sin\theta\right) - X_2 \\ &= m\tau^2 R\left[\left(\frac{n-l}{n} - \frac{c\sqrt{n^2-(\sin\theta-b)^2}}{n(\sin\theta-b)}\right)\left(\cos\theta + \frac{n-l}{n}Z + \frac{c\sin\theta}{n}\right) + \frac{k^2}{n^2}Z\right]. \end{aligned} \quad (32)$$

To determine the accelerating forces in the direction of the axis of Y we have, for the mass at D,

$$y_1 = R[(l-k)\sin\beta + c\cos\beta];$$

from which

$$\frac{d^2y_1}{d\theta^2} = -R\left(\frac{l-k}{n}\sin\theta + \frac{c}{n}Z\right). \quad (33)$$

Similarly, for the mass at E,

$$\frac{d^2y_2}{d\theta^2} = -R\left(\frac{l+k}{n}\sin\theta + \frac{c}{n}Z\right). \quad (34)$$

If F_{dy} and F_{ey} be the Y accelerating forces for the masses D and E respectively, we have from equations (33) and (34)

$$F_{dy} = -\frac{1}{2}m\tau^2 R\left(\frac{l-k}{n}\sin\theta + \frac{c}{n}Z\right), \quad (35)$$

and

$$F_{ey} = -\frac{1}{2}m\tau^2 R\left(\frac{l+k}{n}\sin\theta + \frac{c}{n}Z\right). \quad (36)$$

The component of F_{dy} acting at the crank pin will be

$$F_{dy}A = \frac{l-k-c\tan\beta}{n} F_{dy}$$
$$= -\frac{1}{2} mz^2 R\left[\frac{l-k-c\tan\beta}{n}\left(\frac{l-k}{n}\sin\theta + \frac{c}{n}Z\right)\right]; \quad (37)$$

and of F_{cy},

$$F_{cy}A = \frac{l+k-c\tan\beta}{n} F_{cy}$$
$$= -\frac{1}{2} mz^2 R\left[\frac{l+k-c\tan\beta}{n}\left(\frac{l+k}{n}\sin\theta + \frac{c}{n}Z\right)\right]. \quad (38)$$

The Y component of the accelerating force acting at the crank pin will be equal to

$$F_{dy}A + F_{cy}A,$$

or

$$Y_2 = F_{dy}A + F_{cy}A$$
$$= -mz^2 R\left[\left(\frac{l}{n} - \frac{c(\sin\theta - b)}{n\sqrt{n^2-(\sin\theta-b)^2}}\right)\left(\frac{l}{n}\sin\theta + \frac{c}{n}Z\right) + \frac{k^2}{n^2}\sin\theta\right]. \quad (39)$$

The accelerating force Y_1 acting at the wrist pin may be found by taking the sum of the forces in the Y direction that accelerate the masses D and E as given in (35) and (36) and subtracting Y_2, or

$$Y_1 = F_{dy} + F_{cy} - Y_2$$
$$= -mz^2 R\left(\frac{l}{n}\sin\theta + \frac{c}{n}Z\right) - Y_2$$
$$= -mz^2 R\left[\left(\frac{n-l}{n} + \frac{c(\sin\theta - b)}{n\sqrt{n^2-(\sin\theta-b)^2}}\right)\left(\frac{l}{n}\sin\theta + \frac{c}{n}Z\right) - \frac{k^2}{n^2}\sin\theta\right]. \quad (40)$$

Equations (40), (39), (32), and (31) are the same as (20), (21), (22), and (23); the first and second method therefore give the same results.

RATIO BETWEEN THE LENGTH OF STROKE OF THE PISTON AND THE RADIUS OF THE CRANK.

The length of stroke s will be the distance traveled by the wrist pin between the two positions at which $\theta = -\beta$,

or
$$s = R_1\sqrt{(n+1)^2 - b^2} - R_1\sqrt{(n-1)^2 - b^2}. \tag{41}$$

DETERMINATION OF THE NET FORCES ACTING AT VARIOUS POINTS.

The forces concerned in the action of an engine are the force of the steam on the piston, the forces required to accelerate the parts, and their weight and friction. We will first determine the effect of the pressure of the steam combined with the accelerating forces, omitting the weight and friction of the parts.

Let P_a = effort of the steam on the piston;
G = reaction of cross-head guides;
P_w = force exerted by the connecting rod upon the wrist pin;
P_c = force exerted by the connecting rod upon the crank pin;
R_1 = resultant in direction of the centre line of the rod;*
T = tangential component of P_c;
N = radial component of P_c;
H_c = force exerted by the bed on the crank shaft.

The force exerted by the steam on the piston is equal to the difference of the forces exerted on the two cylinder heads, or to the force acting on the cylinder as a whole; its equilibrium may, therefore, be represented by two lines representing the pressure in magnitude and direction and directly opposed to each other, as shown in Fig. 4, the resultant of which will be zero.

The equilibrium of the piston, piston-rod, and cross-head (see P, Fig. 3) is represented in Fig. 5.

From the pressure of the steam on the piston there must be subtracted the force F_1 required to accelerate the mass of the piston, piston-rod, and

* In our analysis we have determined the forces at the wrist and crank pins that will be equivalent in accelerating effect to the summation of the elementary accelerating forces acting at all portions of the mass of the rod, but these components cannot be used to obtain the internal strains at each portion of the rod, unless the law of the forces acting at the separate elements of its mass is included in the problem. R_1 does not, therefore, represent the actual force along the line of centres of the rod, but simply the resultant in that direction that must be common to the polygons of equilibrium at each of the pins. R_1 will have a variety of values, according to the choice of the forces to give the accelerating moment; but it is a principle in mechanics that the external forces cannot be altered by this means.

cross-head, in order to determine the force in the direction of the motion of the piston that acts on the wrist pin; let this force be F, and it follows that

$$F = P_a - F_1. \tag{42}$$

The equilibrium of the wrist pin (see Q, Fig. 3) is represented in Fig. 6. The forces transmitted from the cross-head to the wrist pin are the horizontal force F and the reaction of the guides G, and these are in equilibrium with the force P_w.

From Fig. 6 it follows that

$$P_w = \sqrt{(P_a - F_1)^2 + G^2}. \tag{43}$$

Let the angle made by P_w with the axis of X be γ; then

$$P_w = F \sec \gamma = (P_a - F_1) \sec \gamma. \tag{44}$$

The equilibrium of the wrist-pin box (see q, Fig. 3) is represented in Fig. 7. The force P_w is in equilibrium with the force R_1 in the direction of the line of centres of the connecting rod and the accelerating force at the wrist pin. The accelerating force at the wrist pin is the resultant of its two components X_1 and Y_1, and these components, and not the force itself, are represented in the figure.

From Fig. 7 we have

$$R_1 = (P_w \cos \gamma - X_1) \sec \beta. \tag{45}$$

The equilibrium of the crank-pin box (see s, Fig. 3) is represented in Fig. 8. X_2 and Y_2, together with R_1, are in equilibrium with the pressure P_c. From the figure we have

$$\begin{aligned} P_c &= \sqrt{[(R_1 \cos \beta - X_2)^2 + (R_1 \sin \beta - Y_2)^2]} \\ &= \sqrt{\{(P_a - F_1 - X_1 - X_2)^2 + [(P_a - F_1 - X_1) \tan \beta - Y_2]^2\}}. \end{aligned} \tag{46}$$

Let the angle made by P_c with the axis of X be ω; then

$$P_c \cos \omega = R_1 \cos \beta - X_2. \tag{47}$$

The equilibrium of the crank pin (see S, Fig. 3) is represented in Fig. 9.

From Fig. 9 we have

$$\begin{aligned} T &= P_c \cos (90 - \theta + \omega) \\ &= P_c (\sin \theta \cos \omega + \cos \theta \sin \omega). \end{aligned}$$

From Fig. 8

$$P_c \sin \omega = R_1 \sin \beta - Y_2. \tag{48}$$

On introducing the values given in (47), (48), (45), and (44), in the above equation for T, we have

$$T = (P_a - F_1 - X_1 - X_2)\sin\theta - Y_2\cos\theta + (P_a - F_1 - X_1)\tan\beta\cos\theta;$$

from which

$$T = (P_a - F_1 - X_1)\sec\beta\sin(\theta+\beta) - Y_2\cos\theta - X_2\sin\theta. \qquad (49)$$

We may obtain, in a similar manner,

$$N = (P_a - F - X_1)\sec\beta\cos(\theta+\beta) + Y_2\sin\theta - X_2\cos\theta. \qquad (50)$$

It also follows from Fig. 9 that

$$P_c = \sqrt{T^2 + N^2}. \qquad (51)$$

The equilibrium of the crank shaft is represented in Fig. 10. The force P_c acting on the crank pin will be transmitted to the crank shaft, and this force will be in equilibrium with the pressure on the crank-shaft bearings H_c, and the weight of the fly-wheel, and the force exerted on the shaft by the mechanism that transmits the power developed by the engine, the sum of which we will call H_1. Let φ be the angle made by H_1 with the axis of X, and μ that made by H_c; then from Fig. 10

$$H_c\cos\mu = P_c\cos\omega + H_1\cos\varphi.$$

Introducing values given in (48) and (45), this becomes

$$H_c = (P_a - F - X_1 - X_2)\sec\mu + H_1\frac{\cos\varphi}{\cos\mu}. \qquad (52)$$

The equilibrium of the bed of the engine (see B, Fig. 3) is represented in Fig. 11. The forces acting on the bed are the force P_a at the cylinder head, the reaction of the guides G, the force H_c at the crank-shaft journals, and the weight of the bed, which we will call B_w, acting at the angle δ with the axis of Y. These will be in equilibrium with the force between the foundation and the bed of the engine. Let this force be H_b, and the angle it makes with the axis of X be ζ; then it follows from Fig. 11 that

$$H_b\cos\zeta = P_a - H_c\cos\mu - B_w\sin\delta;$$

from which

$$H_b = (P_a - H_c\cos\mu - B_w\sin\delta)\sec\zeta. \qquad (53)$$

The following value of H_b may also be obtained:

$$H_b = \text{I}\ [(F_1 + X_1 + X_2 - H_1 \cos \varphi - B_w \sin \delta)^2 + (B_w \cos \delta + G + H_c \sin \mu)^2]. \quad (54)$$

From Fig. 10 we have

$$H_c \sin \mu = H_1 \sin \varphi - P_c \sin \omega,$$

which substituted in equation (53) gives

$$H_b = \text{I}\ [(F_1 + X_1 + X_2 - H_1 \cos \varphi - B_w \sin \delta)^2 + (B_w \cos \delta + G + H_1 \sin \varphi - P_c \sin \omega)^2].$$

From Figs. 5, 7, and 8 we obtain

$$P_c \sin \omega = G - Y_1 - Y_2,$$

which reduces the above value of H_b to

$$H_b = \text{I}\ [(F_1 + X_1 + X_2 - H_1 \cos \varphi - B_w \sin \delta)^2 + (H_1 \sin \varphi + B_w \cos \delta + Y_1 + Y_2)^2]. \quad (55)$$

The X component of H_b, which we will call H_{bx}, is

$$H_{bx} = H_b \cos \zeta.$$

Substituting the value of H_b as given in (53), this becomes

$$H_{bx} = F_1 + X_1 + X_2 - H_1 \cos \varphi - B_w \sin \delta.$$

It follows from (55) that the Y component will be

$$H_{by} = Y_1 + Y_2 + H_1 \sin \varphi + B_w \cos \delta.$$

If X_t and Y_t are the components of the translative force tending to shake the bed, they will act in an opposite direction to the forces that act from the foundation to the bed. As a constant force does not tend to produce a shake, X_t and Y_t will only contain that portion of H_{bx} and H_{by} that is variable. The weight of the foundation B_w will be constant, and, provided the method of transmitting power from the engine does not alter the value of H_1, which is ordinarily very nearly so, its value may also be considered as constant, and we will have

$$X_t = - F_1 - X_1 - X_2, \quad (56)$$

and

$$Y_t = - Y_1 - Y_2. \quad (57)$$

These forces, as has already been stated, tend to translate the bed of the engine, and do not indicate the value of the moment that may be produced by forces opposite in direction and not having the same line of action.

Figs. 4–11 may be combined, and a polygon formed that represents the equilibrium of the entire engine, as shown in Fig. 12.

All the equations already given for the values of the forces may be derived from Fig. 12.

Equation (49) may be written

$$T = \left[P_a - \left(F_1 + X_1 + \frac{Y_2 \cos\theta + X_2 \sin\theta}{\sec\beta \sin(\theta+\beta)} \right) \right] \sec\beta \sin(\theta+\beta).$$

The expression

$$F_1 + X_1 + \frac{Y_2 \cos\theta + X_2 \sin\theta}{\sec\beta \sin(\theta+\beta)},$$

which we will call P_b, therefore represents a pressure which on being subtracted from P_a will give a force that may be resolved as if no motion were present in order to obtain T, or

$$T = (P_a - P_b) \sec\beta \sin(\theta+\beta). \tag{58}$$

P_b may be laid off on the indicator diagram, and its value thus subtracted directly from the steam pressure.

FRICTION AND GRAVITY INCLUDED.

It has been demonstrated, in a paper prepared by Professor Webb and the writer,* that the effect of the friction at the connecting-rod bearings may be determined by introducing into the equilibrium polygons two forces, A and B, at right angles to the centre line of the rod, the sum of A and B being applied at each of the pins in opposite directions. The values of A and B are

$$A = \frac{P_w f r_w \sin\varphi_w}{nR}, \quad \text{and} \quad B = \frac{P_c f r_c \sin\varphi_c}{nR},$$

in which r_w and r_c are equal, respectively, to the radii of the wrist and crank, and $\sin\varphi_w$ and $\sin\varphi_c$ to the sines of the angles of friction. The effect of the weight was also included, and equilibrium polygons given showing the relation of the forces at the wrist and crank pins, together with equations deduced from the same.

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ON THE CONSTRUCTION OF THE PARABOLAS GIVEN BY FOUR POINTS.

By Prof. E. W. Hyde, Cincinnati, Ohio.

Salmon, in his Conic Sections, gives several solutions of the problem of constructing a conic when four points and a tangent are given, which of course includes the above problem, since the line at ∞ in the plane of the curve is a tangent to it. His solutions, however, depend on Carnot's theorem, or the use of the properties of points in involution.

The following method is a direct and simple application of Pascal's theorem, which recently occurred to me. It consists in finding the direction of the double point at ∞; that is, of the axis of the parabola. There are two solutions, as will appear. Let the given points be 1, 2, 3, 4, and call the point and line at ∞ p_x and L_x, respectively. Then the sides of the Pascal hexagon may be taken as $\begin{matrix}12\\4p_x\end{matrix}$, $\begin{matrix}23\\L_x\end{matrix}$, $\begin{matrix}34\\p_x1\end{matrix}$, in order; and the three points $(12\,.\,4p_x)$, $(23\,.\,L_x)$, $(34\,.\,1p_x)$ must lie on one line, say A. Hence we have to draw through 1 and 4, respectively, two parallel lines, $1p_x$ and $4p_x$, cutting 34 and 12 in points 6 and 7, such that 67, *i. e.* A, shall be parallel to 23. To find the point 6, let $A3 = a'$, $A4 = a$, $A2 = b'$, $A1 = b$, $A5 = a_1$, $A6 = x$, and $A7 = y$. Then by similar triangles we must have

$$\frac{x}{y} = \frac{a'}{b'} \quad \text{and} \quad \frac{x}{a} = \frac{b}{y};$$

whence

$$x = \pm\sqrt{a \cdot \frac{ba'}{b'}} = \pm\sqrt{aa_1}.$$

The distance $a_1 = A5$ is constructed at once by drawing 15 parallel to 23; then draw a circle with diameter $A4$, and erect at 5 a perpendicular to $A4$, cutting the circle in 9; with $A9$ as radius, draw circle cutting $A4$ in $A6\,(= x)$ and $A8\,(= -x)$; then 16 and 18 are parallel to the axes of the two parabolas. Having the point at ∞ for each parabola, we have five points, and can apply Pascal's theorem in the usual way to construct the curve.

ON THE METHOD OF CONTINUED IDENTITY.*

By Mr. Chas. H. Kummell, Washington, D. C.

This apparently new method of expressing a quantity is based on the following considerations :—

Suppose a quantity x is approximately known by its functional properties, so that we can place

$$x = \frac{1 + x_1}{q}, \tag{1}$$

where $1/q$ is the approximate value of x and x_1/q its correction. Again, assume

$$x_1 = qx - 1 = \frac{1 + x_2}{q_1}, \tag{1$_1$}$$

$$x_2 = q_1 x_1 - 1 = \frac{1 + x_3}{q_2}, \tag{1$_2$}$$

. . . . . .

$$x_n = q_{n-1} x_{n-1} - 1 = \frac{1 + x_{n+1}}{q_n}; \tag{1$_n$}$$

then we shall have

$$x = \frac{1}{q}\left[1 + \frac{1}{q_1}\left[1 + \frac{1}{q_2}\left[1 + \ldots\right.\right.\right. \tag{2}$$

$$= \frac{x}{1 + x_1}\left[1 + \frac{x_1}{1 + x_2}\left[1 + \frac{x_2}{1 + x_3}\left[1 + \ldots .\right.\right.\right. \tag{3}$$

Suppose any one x, for instance x_3, $= 0$, or sufficiently nearly so; then we have

$$x = \frac{x}{1 + x_1}\left[1 + \frac{x_1}{1 + x_2}(1 + x_2)\right]$$

$$= \frac{x}{1 + x_1}(1 + x_1)$$

$$= x, \quad \text{a perfect identity;}$$

and it is important to notice that this identity holds, whatever the preceding values of x_2 and x_1 may have been, if only $x_3 = 0$ or is sufficiently small to be

* Read before the Mathematical Section of the Washington Philosophical Society.

neglected. We may thus continue the identity to any number of steps before perfect or sufficiently perfect identity is reached. Since

$$q = \frac{1 + x_1}{x} = \frac{1}{x}, \quad \text{approximately,}$$
$$q_1 = \frac{1 + x_2}{x_1} = \frac{1}{x_1} \quad \text{“}$$
$$q_2 = \frac{1 + x_3}{x_2} = \frac{1}{x_2} \quad \text{“}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$q_n = \frac{1 + x_{n+1}}{x_n} = \frac{1}{x_n} \quad \text{“}$$

the condition for perfect identity of (2) is that the q's must have ∞ for their limit. Take, for instance, the binomial series which may be written

$$(1 + x)^m = 1 + \frac{m}{1} x \Big[1 + \frac{m-1}{2} x \Big[1 + \frac{m-2}{3} x \Big[1 + \ldots ; \qquad (4)$$

comparing which with (2), we have

$$q = 1, \qquad (5)$$
$$q_1 = \frac{1}{mx}, \qquad (5_1)$$
$$q_2 = \frac{2}{(m-1)x}, \qquad (5_2)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$q_n = \frac{n}{(m-n+1)x}. \qquad (5_n)$$

The condition for perfect identity is then

$$q_n = \infty ; \qquad \therefore m - n + 1 = 0.$$

It follows, then, that the binomial series can give perfect identity only if m is an integer, and that this is attained by $m + 1$ terms. If m is not an integer, perfect identity is impossible, since q_n can never be infinite; and this is also the case if the condition of convergence $x < 1$ is fulfilled. Since, then,

$$q_\infty = \frac{n}{(m-n+1)x} \Big]_{n=\infty} = -\frac{1}{x} > -1,$$

the terms at infinity form a geometric series with the exponent $-x$, which therefore has a sum. It follows, then, that the form of continued identity is

still convergent if the denominators q tend to a limit > 1. This condition holding, we can give an approximate value of the remainder if we stop at any term; for if q_n is sufficiently large, the next term, though small, will exceed the sum of all following terms; and since q_{n+1} must be at least $= q_n$, we have

$$(n+1)\text{th term} < \frac{1}{q} \cdot \frac{1}{q_1} \cdot \frac{1}{q_2} \cdot \ldots \frac{1}{q_n} \cdot \frac{1}{q_n} = \text{approximate remainder.} \quad (6)$$

Transforming any equation for one unknown quantity x by relations (1) into an equation for $x_1, x_2, \ldots$, we have a method of approximation which is convenient and, in general, rapidly convergent. Suppose we have to solve

$$0 = f(x). \quad (7)$$

Let

$$x = x_0 + x_1, \quad (8)$$

where x_0 is the integral and x_1 the fractional part of the root. By Taylor's theorem we have

$$0 = f(x_0) + f'(x_0)\,x_1 + \frac{f''(x_0)}{2!}\,x_1^2 + \frac{f'''(x_0)}{3!}\,x_1^3 + \ldots. \quad (9)$$

Let

$$x_1 = \frac{1 + x_2}{q_1}, \quad (1_1)$$

where q is the integral part of $-\dfrac{f'(x_0)}{f'(x_0)}$ (preferably exceeding it). To transform (9) to an equation in x_2 we have to decide first to which order of precision we wish to satisfy the equation. To fix ideas, let us suppose the solution to be exact to the 3d order; we have then to solve a cubic equation, which we shall write thus:

$$0 = a_1''' + a_1''x_1 + a_1'x_1^2 + a_1x_1^3. \quad (9_1)$$

Transforming this to one having its roots q_1 times as great, we have

$$0 = q_1^3a_1''' + q_1^2a_1''(q_1x_1) + q_1a_1'(q_1x_1)^2 + a_1(q_1x_1)^3;$$

then transforming this to one having its roots diminished by 1, we have

$$\begin{aligned} 0 &= a_1 + a_1'q_1 + a_1''q_1^2 + a_1'''q^3 \\ &\quad + (3a_1 + 2a_1'q_1 + a_1''q_1^2)\,x_2 + (3a_1 + a_1'q_1)\,x_2^2 + a_1x_2^3 \\ &= a_2''' + a_2''x_2 + a_2'x_2^2 + a_2x_2^3. \end{aligned} \quad (9_2)$$

Let

$$x_2 = \frac{1 + x_3}{q_2}, \quad (1_2)$$

where q_2 = integral part of $-\frac{a_2''}{a_2'''}$ (preferably exceeding it). Transforming to $q_2 x_2 - 1$ as before, we have

$$0 = a_3''' + a_3'' x_3 + a_3' x_3^2 + a_3 x_3^3.$$

This process is continued as far as we please, in case the cubic (9_1) is proposed for solution. If, however, (9_1) coincides with the complete equation (9) to the 3d order only, it will be useless to make more than three transformations. A convenient form for making these transformations I shall exhibit in the solution of the cubic

$$0 = 3 - 5x + x^3.$$

This equation has three real roots, one between 0 and 1, another between 1 and 2, and the third between -2 and -3. Here $\frac{3}{5}$ is an approximate value for the smallest root; hence $q = 2 > \frac{3}{5}$, and a more exact value of the root is found by the following algorithm:

$$\begin{array}{rrrrrrl}
0 = & + & 3 & - & 5x & + 0x^2 & + x^3 \qquad x = \dfrac{1+x_1}{2} \\
& + & 6 & - & 10 & 0 & + 1 \\
& + & 12 & - & 20 & + 1 & + 1 \\
& + & 24 & - & 19 & + 2 & + 1 \\
0 = & + & 5 & - & 17x_1 & + 3x_1^2 & + x_1^3 \qquad x_1 = \dfrac{1+x_2}{4} \\
& + & 20 & - & 68 & + 12 & + 1 \\
& + & 80 & - & 272 & + 13 & + 1 \\
& + & 320 & - & 259 & + 14 & + 1 \\
0 = & + & 61 & - & 245x_2 & + 15x_2^2 & + x_2^3 \qquad x_2 = \dfrac{1+x_3}{4} \\
& + & 244 & - & 980 & + 60 & + 1 \\
& + & 976 & - & 3920 & + 61 & + 1 \\
& + & 3904 & - & 3859 & + 62 & + 1 \\
0 = & + & 45 & - & 3797x_3 & + 63x_3^2 & + x_3^3 \qquad x_3 = \dfrac{1+x_4}{84}
\end{array}$$

The root so far determined is correct within less than $\frac{1}{2 \times 4 \times 4 \times 84^2} = \frac{1}{86016}$. We have, then, neglecting x_4,

$$\begin{aligned} x_3 &= 0.011905, \\ x_2 &= 0.252976, \\ x_1 &= 0.313244, \\ x &= 0.656622. \end{aligned}$$

The correct value is $x = 0.656620.$

To determine another root, for instance that lying between -2 and -3, we may first increase the roots by 2 by assuming $x = y - 2$; then y will lie between 0 and -1, and the more exact value is found thus:

$$
\begin{array}{rrrrrrrrl}
0 = & + & 5 & + & 7y & - & 6y^2 & + y^3 & y = \dfrac{1 + y_1}{-2} \\
 & - & 10 & - & 14 & + & 12 & + 1 & \\
 & + & 20 & + & 28 & + & 13 & + 1 & \\
 & - & 40 & + & 41 & + & 14 & + 1 & \\
0 = & + & 1 & + & 55y_1 & + & 15{y_1}^2 & + {y_1}^3 & y_1 = \dfrac{1 + y_2}{-55} \\
 & - & 55 & - & 3025 & - & 825 & + 1 & \\
 & + & 3025 & + & 166375 & - & 824 & + 1 & \\
 & - & 166375 & + & 165551 & - & 823 & + 1 & \\
0 = & - & 824 & + & 164728y_2 & - & 822{y_2}^2 & + {y_2}^3 & y_2 = \dfrac{1 + y_3}{200}
\end{array}
$$

This determination must be correct within less than $\dfrac{1}{2 \times 55 \times 200^2} = \dfrac{1}{4400000}$, and we have

$$
\begin{aligned}
y_2 &= +0.005, \\
y_1 &= -0.0182727, \\
y &= -0.4908636, \\
x &= -2.4908636.
\end{aligned}
$$

The correct value is $x = -2.4908638$.

We notice the greater convergence in the computation of this root. This is caused by its being more different from the other roots.

Should two or more roots be very close together, the process becomes very slowly convergent. This is, indeed, a difficulty common to all numerical methods of solution of higher equations. It may be in most cases sufficiently overcome by placing the nearly equal roots between 0 and 1, and then determine their reciprocals. For a test of this suggestion, I solve the equation,

$$0 = 7 - 7x + x^3,$$

which has two roots between $+1$ and $+2$. Here the direct process gives

$$
\begin{aligned}
x &= \frac{1}{1}\left[1 + \frac{1}{4}\left[1 + \frac{1}{3}\left[1 + \frac{1}{3}\left[1 - \frac{1}{7}\left[1 + \frac{1}{16}\left[1 + \ldots\right.\right.\right.\right.\right.\right. \\
&= 1.3568948.
\end{aligned}
$$

If we assume $x = y + 1$, then

$$0 = 1 - 4y + 3y^2 + y^3;$$

and placing $y = z^{-1}$, we have

$$0 = 1 + 3z - 4z^2 + z^3.$$

This equation has one positive root between $+1$ and $+2$, and one between $+2$ and $+3$. To determine the larger, which corresponds to the root already given, we assume $z = v + 3$, and have

$$\begin{array}{rrrrrl}
0 = + & 1 & + 6v & + 5v^2 & + v^3 & v = \dfrac{1 + v_1}{-6} \\
 & - 6 & - 36 & - 30 & + 1 & \\
 & + 36 & + 216 & - 29 & + 1 & \\
 & - 216 & + 187 & - 28 & + 1 & \\
0 = & - 29 & + 159v_1 & - 27{v_1}^2 & + {v_1}^3 & v_1 = \dfrac{1 + v_2}{5} \\
 & - 145 & + 795 & - 135 & + 1 & \\
 & - 725 & + 3975 & - 134 & + 1 & \\
 & - 3625 & + 3841 & - 133 & + 1 & \\
0 = + & 216 & + 3708v_2 & - 132{v_2}^2 & + {v_2}^3 & v_2 = \dfrac{1 + v_3}{-17}
\end{array}$$

This should give the root within less than $\dfrac{1}{6 \times 5 \times 17^2} = \dfrac{1}{8670}$, and we have

$$\begin{aligned}
v_2 &= -0.05882, \\
v_1 &= +0.18824, \\
v &= -0.19804, \\
z &= +2.80196, \\
y &= +0.35689, \\
x &= +1.35689.
\end{aligned}$$

The exact value is $x = +1.356896.$

If we apply the algorithm to the equation

$$0 = 8 - 7x + x^3,$$

which differs from the above equation by 1 in the numerical term, the two roots corresponding to those before nearly equal will be imaginary, and we have

$$
\begin{array}{rrrrrrl}
0 = & + 8 & - 7x & + 0x^2 & + x^3 & x = \dfrac{1+x_1}{1} \\
& + 8 & - 7 & + 0 & + 1 & \\
& + 8 & - 7 & + 1 & + 1 & \\
& + 8 & - 6 & + 2 & + 1 & \\
0 = & + 2 & - 4x_1 & + 3x_1^2 & + x_1^3 & x_1 = \dfrac{1+x_2}{2} \\
& + 4 & - 8 & + 6 & + 1 & \\
& + 8 & - 16 & + 7 & + 1 & \\
& + 16 & - 9 & + 8 & + 1 & \\
0 = & + 7 & - 1x_2 & + 9x_2^2 & + x_2^3 & x_2 = \dfrac{1+x_3}{1} \\
& + 7 & - 1 & + 9 & + 1 & \\
& + 7 & - 1 & + 10 & + 1 & \\
& + 7 & + 9 & + 11 & + 1 & \\
0 = & + 16 & + 20x_3 & + 12x_3^2 & + x_3^3 & x_3 = \dfrac{1+x_4}{-2} \\
& - 32 & - 40 & - 24 & + 1 & \\
& + 64 & + 80 & - 23 & + 1 & \\
& - 128 & + 57 & - 22 & + 1 & \\
0 = & - 71 & + 35x_4 & - 21x_4^2 & + x_4^3 & x_4 = \dfrac{1+x_5}{1} \\
& - 71 & + 35 & - 21 & + 1 & \\
& - 71 & + 35 & - 20 & + 1 & \\
& - 71 & + 15 & - 19 & + 1 & \\
0 = & - 56 & - 4x_5 & - 18x_5^2 & + x_5^3 & x_5 = \dfrac{1+x_6}{-1} \\
& + 56 & + 4 & + 18 & + 1 & \\
& - 56 & - 4 & + 19 & + 1 & \\
& + 56 & + 15 & + 20 & + 1 & \\
0 = & + 71 & + 35x_6 & + 21x_6^2 & + x_6^3 & x_6 = \dfrac{1+x_7}{-1}
\end{array}
$$

We have, therefore, the periodic form

$$x = \frac{1}{1}\left[1+\frac{1}{2}\left[1+\frac{1}{1}\left[1-\frac{1}{2}\left[1+\frac{1}{1}\left[1-\frac{1}{1}\left[1-\frac{1}{1}\left[1+\frac{1}{1}\left[1-\frac{1}{1}\left[1-\frac{1}{1}\left[1+\ldots,\right.\right.\right.\right.\right.\right.\right.\right.\right.$$

which obviously has no value.

The application of the method to the general quadratic

$$0 = a'' + a'x + x^2$$

is of great interest. Assuming $x = \dfrac{1+x_1}{-a'a''^{-1}}$, we have the transformed equation in x_1

$$0 = 1 + (2 - a'^2 a''^{-1})\, x_1 + x_1^2.$$

Assuming

$$x_1 = \frac{1 + x_2}{a'^2 a''^{-1} - 2},$$

we have the equation in x_2

$$0 = 1 + [2 - (a'^2 a''^{-1} - 2)^2]\, x_2 + {x_2}^2.$$

We have, therefore,

$$x = -\frac{1}{a' a''^{-1}}\left[1 + \frac{1}{a'^2 a''^{-1} - 2}\left[1 + \frac{1}{(a'^2 a''^{-1} - 2)^2 - 2}\left[1 + \frac{1}{[(a'^2 a''^{-1} - 2)^2 - 2]^2 - 2}\left[1 + \ldots, \quad (10)\right.\right.\right.\right.$$

or if we place

$$\begin{aligned} q &= -a' a''^{-1}, \\ q_1 &= -a'^2 a''^{-1} - 2, \end{aligned}$$

then will

$$\begin{aligned} q_2 &= {q_1}^2 - 2, \\ q_3 &= {q_2}^2 - 2, \\ &\cdots \end{aligned} \qquad (11)$$

We have

$$x = -\frac{a'}{2} + \sqrt{\frac{a'^2}{4} - a''};$$

therefore,

$$\sqrt{\frac{a'^2}{4} - a''} = \frac{a'}{2} - \frac{a''}{a'}\left[1 + \frac{1}{a'^2 a''^{-1} - 2}\left[1 + \frac{1}{{q_1}^2 - 2}\left[1 + \frac{1}{{q_2}^2 - 2}\left[1 + \ldots . \quad (12)\right.\right.\right.\right.$$

Placing

$$z = -\frac{4a''}{a'^2}, \qquad (13)$$

and dividing by $\frac{a'}{2}$,

$$\sqrt{1 + z} = 1 + \frac{z}{2}\left[1 - \frac{1}{4z^{-1} + 2}\left[1 + \frac{1}{16z^{-2} + 16z^{-1} + 2}\left[1 + \ldots . \quad (14)\right.\right.\right.$$

Now this form, which is essentially Robert's rule for extracting a square root,* has the remarkable property of being *convergent for any value of z*, while its value by the binomial series

$$\sqrt{1 + z} = 1 + \frac{1}{2}z\left[1 - \frac{1}{4}z\left[1 - \frac{3}{6}z\left[1 - \frac{5}{8}z\left[1 - \ldots\right.\right.\right.\right.$$

* Communicated to the Mathematical Section of the Philosophical Society of Washington by Dr. A. Martin, and published in the Mathematical Magazine, Vol. II, p. 13, as problem 132.

ceases to be convergent if $z > 1$; for the denominators q, q_1, q_2, . . . can never be < 2, whatever the value of z. If, however, $z < 1$, these denominators tend to infinity, which will bring about perfect identity, while the form is only convergent if $z > 1$.

If we assume

$$f(x) = \frac{1}{q}\left[1 + \frac{1}{q_1 x^{-1} + q_1'}\right[1 + \frac{1}{q_2 x^{-2} + q_2' x^{-1} + q_2''}\left[1 + \frac{1}{q_3 x^{-3} + q_3' x^{-2} + q_3'' x^{-1} + q_3'''}\right[1 + \ldots, \quad (15)$$

we have a form which becomes perfectly identical if $x < 1$, and is convergent only if $x > 1$, provided the quantities q, q_1', q_2'', q_3''', . . . tend to values > 1.

We may assume identically

$$f(x) = \frac{f(x)}{1 + f_1(x)}\left[1 + \frac{f_1(x)}{1 + f_2(x)}\right[1 + \frac{f_2(x)}{1 + f_3(x)}\left[1 + \ldots, \quad (16)$$

so that

$$\begin{aligned}
q &= \frac{1 + f_1(x)}{f(x)}, \\
q_1 x^{-1} + q_1' &= \frac{1 + f_2(x)}{f_1(x)}, \\
q_2 x^{-2} + q_2' x^{-1} + q_2'' &= \frac{1 + f_3(x)}{f_2(x)}, \qquad (17) \\
q_3 x^{-3} + q_3' x^{-2} + q_3'' x^{-1} + q_3''' &= \frac{1 + f_4(x)}{f_3(x)}, \\
&\ldots\ldots\ldots\ldots
\end{aligned}$$

and by McLaurin's theorem we have

$$\begin{aligned}
f_1(x) &= f_1(0) + f_1'(0) \cdot x + \frac{f_1''(0)}{2!} x^2 + \ldots \\
&= -1 + q\left[f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} x^2 + \ldots\right];
\end{aligned}$$

hence, if $f_1(0) = 0$,

$$\begin{aligned}
0 = f_1(0) &= -1 + q f(0); \quad \therefore q = \frac{1}{f(0)}, \qquad (18) \\
f_1'(0) &= q f'(0), \\
\frac{f_1''(0)}{2!} &= q \frac{f''(0)}{2!}, \qquad (19_1) \\
&\ldots\ldots
\end{aligned}$$

We have further, if $f_2(0) = f_2'(0) = 0$,

$$f_2(x) = \frac{f_2''(0)}{2!}x^2 + \frac{f_2'''(0)}{3!}x^3 + \ldots$$
$$= -1 + (q_1 + q_1'x)\left[f_1'(0) + \frac{f_1''(0)}{2!}x + \frac{f_1'''(0)}{3!}x^2 + \ldots\right];$$

hence

$$0 = f_2(0) = -1 + q_1 f_1'(0); \quad \therefore q_1 = \frac{1}{f_1'(0)}, \tag{18_1}$$

$$0 = f_2'(0) = q_1\frac{f_1''(0)}{2!} + q_1' f_1'(0); \quad \therefore q_1' = -\frac{1}{f_1'(0)}q_1\frac{f_1''(0)}{2!}, \tag{$18_1'$}$$

$$\frac{f_2''(0)}{2!} = q_1\frac{f_1'''(0)}{3!} + q_1'\frac{f_1''(0)}{2!},$$

$$\frac{f_2'''(0)}{3!} = q_1\frac{f_1^{\mathrm{iv}}(0)}{4!} + q_1'\frac{f_1'''(0)}{3!}, \tag{19_2}$$

. . . . . .

$$0 = f_3(0) = -1 + q_2\frac{f_2''(0)}{2!}; \quad \therefore q_2 = \frac{2!}{f_2''(0)}, \tag{18_2}$$

$$0 = f_3'(0) = q_2\frac{f_2'''(0)}{3!} + q_2'\frac{f_2''(0)}{2!}; \quad \therefore q_2' = -\frac{2!}{f_2''(0)}q_2\frac{f_2'''(0)}{3!}, \tag{$18_2'$}$$

$$0 = \frac{f_3''(0)}{2!} = q_2\frac{f_2^{\mathrm{iv}}(0)}{4!} + q_2'\frac{f_2'''(0)}{3!} + q_2''\frac{f_2''(0)}{2!};$$
$$\therefore q_2'' = -\frac{2!}{f_2''(0)}\left[q_2\frac{f_2^{\mathrm{iv}}(0)}{4!} + q_2'\frac{f_2'''(0)}{3!}\right], \tag{$18_2''$}$$

$$\frac{f_3'''(0)}{3!} = q_2\frac{f_2^{\mathrm{v}}(0)}{5!} + q_2'\frac{f_2^{\mathrm{iv}}(0)}{4!} + q_2''\frac{f_2'''(0)}{3!},$$

$$\frac{f_3^{\mathrm{iv}}(0)}{4!} = q_2\frac{f_2^{\mathrm{vi}}(0)}{6!} + q_2'\frac{f_2^{\mathrm{v}}(0)}{5!} + q_2''\frac{f_2^{\mathrm{iv}}(0)}{4!}, \tag{19_3}$$

. . . . . .

and so on.

If the proposed function $f(x)$ vanishes for $x = 0$ by virtue of a factor x^p (where p may be either whole or fractional), we apply the above process to the function

$$x^{-p}f(x), \text{ denoting it } f(x),$$

or to

$$x^{-p+1}f(x), \quad \text{“} \quad \text{“} \quad f_1(x),$$
$$x^{-p+2}f(x), \quad \text{“} \quad \text{“} \quad f_2(x),$$
. . . . . .

If the function is par, or $f(x) = f(-x)$, its development contains only even powers of x, and placing $x^2 = z$ we have the required form. If the

function is impar, or $f(x) = -f(-x)$, dividing by x and placing $x^2 = z$ we have again the required form.

We may also arbitrarily modify the complete form (15), if, for instance, we do not determine all the q's at the nth term. This, however, changes nothing essential in the rule in equations (18) and (19). At any step there will be as many equations (18) as we choose; then formulæ (19) follow where (18) have been interrupted. To explain this modification, let us suppose we only want two q's in the fourth term, then we have

$$0 = f_4(0) = -1 + q_3 \frac{f_3'''(0)}{3!}; \quad \therefore q_3 = \frac{3!}{f_3'''(0)}, \tag{18$_3$}$$

$$0 = f_4'(0) = q_3 \frac{f_3^{\text{iv}}(0)}{4!} + q_3' \frac{f_3'''(0)}{3!}; \quad \therefore q_3' = -\frac{3!}{f_3'''(0)} q_3 \frac{f_3^{\text{iv}}(0)}{4!}, \tag{18$_3'$}$$

$$\frac{f_4''(0)}{2!} = q_3 \frac{f_3^{\text{v}}(0)}{5!} + q_3' \frac{f_3^{\text{iv}}(0)}{4!},$$

$$\frac{f_4'''(0)}{3!} = q_3 \frac{f_3^{\text{vi}}(0)}{6!} + q_3' \frac{f_3^{\text{v}}(0)}{5!}, \tag{19}$$

. . . . .

It happens in some cases, much to the advantage of the form (15), that one or more of equations (19) vanish. In that case a corresponding number of terms in (15) also disappear.

For the first application take the function

$$f(x) = l(1 + x).$$

Since here

$$f(0) = 0 \qquad \text{and} \qquad f'(0) = 1,$$

we put

$$f_1(x) = l(1 + x),$$

so that

$$f_1(0) = 0,$$
$$f_1'(0) = 1,$$
$$\frac{f_1''(0)}{2!} = -\frac{1}{2},$$
$$\frac{f_1'''(0)}{3!} = +\frac{1}{3},$$
. . .
$$\frac{f_1^{(m)}(0)}{m!} = (-)^{m-1} \frac{1}{m}.$$

We have then

$$0 = q_1 f_1'(0) - 1\,; \quad \therefore q_1 = 1\,,$$

$$0 = q_1 \frac{f_1''(0)}{2!} + q_1' f_1'(0)\,; \quad \therefore q_1' = \frac{1}{2}\,,$$

$$\frac{f_2''(0)}{2!} = 1 \times +\frac{1}{3} + \frac{1}{2} \times -\frac{1}{2} = \frac{1}{3 \cdot 2^2} = \frac{1}{12}\,,$$

$$\frac{f_2'''(0)}{3!} = 1 \times -\frac{1}{4} + \frac{1}{2} \times +\frac{1}{3} = \frac{-2}{4 \cdot 2 \cdot 3} = -\frac{1}{12}\,,$$

$$\frac{f_2^{\mathrm{iv}}(0)}{4!} = 1 \times +\frac{1}{5} + \frac{1}{2} \times -\frac{1}{4} = \frac{3}{5 \cdot 2 \cdot 4} = \frac{3}{40}\,,$$

. . . . . .

$$\frac{f_2^{(m)}(0)}{m!} = \left\{1 \times \frac{1}{m} + \frac{1}{2} \times -\frac{1}{m-1}\right\}(-)^{m-1} = \frac{m-1}{2m\,(m-1)}\,(-)^{m-1}\,;$$

and

$$0 = q_2 \frac{f_2''(0)}{2!} - 1\,; \quad \therefore q_2 = 12,$$

$$0 = q_2 \frac{f_2'''(0)}{3!} + q_2' \frac{f_2''(0)}{2!}\,; \quad \therefore q_2' = 12,$$

$$0 = q_2 \frac{f_2^{\mathrm{iv}}(0)}{4!} + q_2' \frac{f_2'''(0)}{3!} + q_2'' \frac{f_2''(0)}{2!}\,; \quad \therefore q_2'' = \frac{6}{5}\,,$$

$$\frac{f_3'''(0)}{3!} = 12 \times \frac{-4}{2 \cdot 5 \cdot 6} + 12 \times \frac{3}{2 \cdot 4 \cdot 5} + \frac{6}{5} \times \frac{-2}{2 \cdot 3 \cdot 4} = 0\,,$$

$$\frac{f_3^{\mathrm{iv}}(0)}{4!} = 12 \times \frac{5}{2 \cdot 6 \cdot 7} + 12 \times \frac{-4}{2 \cdot 5 \cdot 6} + \frac{6}{5} \times \frac{3}{2 \cdot 4 \cdot 5} = \frac{3}{700}\,,$$

$$\frac{f_3^{\mathrm{v}}(0)}{5!} = 12 \times \frac{-6}{2 \cdot 7 \cdot 8} + 12 \times \frac{5}{2 \cdot 6 \cdot 7} + \frac{6}{5} \times \frac{-4}{2 \cdot 5 \cdot 6} = \frac{-3}{350}\,,$$

$$\frac{f_3^{\mathrm{vi}}(0)}{6!} = 12 \times \frac{7}{2 \cdot 8 \cdot 9} + 12 \times \frac{-6}{2 \cdot 7 \cdot 8} + \frac{6}{5} \times \frac{5}{2 \cdot 6 \cdot 7} = \frac{1}{84}\,,$$

$$\frac{f_3^{\mathrm{vii}}(0)}{7!} = 12 \times \frac{-8}{2 \cdot 9 \cdot 10} + 12 \times \frac{7}{2 \cdot 8 \cdot 9} + \frac{6}{5} \times \frac{-5}{2 \cdot 7 \cdot 8} = \frac{1}{70}\,,$$

$$\frac{f_3^{\mathrm{viii}}(0)}{8!} = 12 \times \frac{9}{2 \cdot 10 \cdot 11} + 12 \times \frac{-8}{2 \cdot 9 \cdot 10} + \frac{6}{5} \times \frac{7}{2 \cdot 8 \cdot 9} = \frac{7}{440}\,,$$

$$\frac{f_3^{\mathrm{ix}}(0)}{9!} = 12 \times \frac{-10}{2 \cdot 11 \cdot 12} + 12 \times \frac{9}{2 \cdot 10 \cdot 11} + \frac{6}{5} \times \frac{-8}{2 \cdot 9 \cdot 10} = -\frac{14}{825}\,,$$

$$\frac{f_3^{\mathrm{x}}(0)}{10!} = 12 \times \frac{11}{2 \cdot 12 \cdot 13} + 12 \times \frac{-10}{2 \cdot 11 \cdot 12} + \frac{6}{5} \times \frac{9}{2 \cdot 10 \cdot 11} = \frac{63}{3575}\,,$$

. . . . . .

Since here $\frac{f_3'''(0)}{3!}$ vanishes, the third term then disappears, and we have

$$0 = \frac{3}{700} q_4 - 1\,; \quad \therefore q_4 = \frac{700}{3}\,,$$

$$0 = -\frac{3}{350} q_4 + \frac{3}{700} q_4'\,; \quad \therefore q_4' = \frac{1400}{3}\,,$$

$$0 = \frac{1}{84} q_4 - \frac{3}{350} q_4' + \frac{3}{700} q_4''\,; \quad \therefore q_4'' = \frac{7700}{27}\,,$$

$$0 = -\frac{1}{70} q_4 + \frac{1}{84} q_4' - \frac{3}{350} q_4'' + \frac{3}{700} q_4'''\,; \quad \therefore q_4''' = \frac{1400}{27}\,,$$

$$0 = \frac{7}{440} q_4 - \frac{1}{70} q_4' + \frac{1}{84} q_4'' - \frac{3}{350} q_4''' + \frac{3}{700} q_4^{\text{iv}}\,; \quad \therefore q_4^{\text{iv}} = \frac{2450}{2673}\,,$$

$$\frac{f_5^{\text{v}}(0)}{5!} = -\frac{14}{825}\cdot\frac{700}{3} + \frac{7}{440}\cdot\frac{1400}{3} - \frac{1}{70}\cdot\frac{7700}{27} + \frac{1}{84}\cdot\frac{1400}{27} - \frac{3}{350}\cdot\frac{2450}{2673} = 0\,,$$

$$\frac{f_5^{\text{vi}}(0)}{6!} = \frac{63}{3575}\cdot\frac{700}{3} - \frac{14}{825}\cdot\frac{1400}{3} + \frac{7}{440}\cdot\frac{7700}{27} - \frac{1}{70}\cdot\frac{1400}{27} + \frac{1}{84}\cdot\frac{2450}{2673} = -\frac{10}{104247}\,,$$

etc., etc.

Since here $q_1' = \frac{1}{2}$, $q_2'' = \frac{6}{5}$, $q_4^{\text{iv}} = \frac{2450}{2673}$, . . . do not appear to tend to a value > 1, we cannot assert that the form

$$l(1+x) = \frac{x}{1+\frac{1}{2}x}\left[1 + \frac{x^2}{12 + 12x + \frac{6}{5}x^2}\left[1 + \frac{x^4}{\frac{700}{3} + \frac{1400}{3}x + \frac{7700}{27}x^2 + \frac{1400}{27}x^3 + \frac{2450}{2673}x^4}\left[1 - \ldots\right.\right.\right. \quad (20)$$

is convergent for every value of x. It may, however, be used for values of $x > 1$, to a considerable extent, as long as the denominators q, $q_1 x^{-1} + q_1'$, $q_2 x^{-2} + q_2' x^{-1} + q_2''$, . . . form an ascending series. If $x < 1$, the form is perfectly identical, and the first two terms will be correct to at least six decimal places.

Placing $x = \frac{z}{1-z}$, we have

$$l(1+x) = l\left[1 + \frac{z}{1-z}\right] = -l(1-z)\,;$$

and by (20),

$$l(1+x) = -l(1-z) = \frac{z}{1-\frac{1}{2}z}\left[1+\frac{z^2}{12-12z+\frac{6}{5}z^2}\right[1$$
$$+\frac{z^4}{\frac{700}{3}-\frac{1400}{3}z+\frac{7700}{27}z^2-\frac{1400}{27}z^3+\frac{2450}{2673}z^4}\left[1-\ldots \quad (21)\right.$$

Since $z = \frac{x}{1+x} < 1$, this form is always perfectly identical.

This example is a fair exhibit of the advantages and defects of the method. Although the required coefficients q are determined by a regular algorithmic process, the resulting form is by no means of such a form that it might be continued by induction, except in case of Robert's rule (14) and the formula for the root of a quadratic (10).

ON THE MOTION OF A PLANET, ASSUMING THAT THE VELOCITY OF GRAVITY IS FINITE.

By Prof. Ormond Stone, University of Virginia, Va.

Laplace, in the Mécanique Céleste (Tome X, Art. 22), considers the hypothesis, that gravitation is produced by the impulsion of a fluid toward the centre of the attracting body, and finds that to account for the acceleration of the moon's mean motion the velocity of the gravific fluid must be at least one hundred million times that of light. He adds:

"Il est aisé de voir que l'équation seculaire de la terre, due à la transmission successive de la gravité, n'est qu'un sixieme environ de l'équation correspondante de la lune, et par conséquent, elle est nulle ou insensible."

Although the value of the acceleration employed by Laplace was erroneous, the argument is not affected thereby.

Lehmann-Filhés (Astronomische Nachrichten, No. 2630) considers the hypothesis that gravity has a finite velocity of propagation; but his hypothesis differs from that of Laplace in that no fluid is assumed. I propose in this note to consider the same hypothesis, retaining the terms in λm which he neglects, but assuming that no secular disturbances are produced by the motion of the solar system through space. Let ξ_0, η_0 and ξ, η be the rectangular co-ordinates of the sun and the planet, in the plane of the orbit, referred to fixed axes. Also put

$$r = \sqrt{(\xi - \xi_0)^2 + (\eta - \eta_0)^2} = \sqrt{x^2 + y^2},$$

in which r is the distance between the two bodies, and

$$x = \xi - \xi_0, \qquad y = \eta - \eta_0$$

are the co-ordinates of the planet with reference to the sun.

Let m be the mass of the planet, taking the sun's mass as the unit.

If we assume that gravity requires an interval $\lambda(1 + m)r$ to pass from the sun to the planet and *vice versa*, the co-ordinates of the sun at the time the impulse starts therefrom are, to terms of the first order,

$$\xi_0 - \lambda (1 + m)\, r \frac{d\xi_0}{dt}, \qquad \eta_0 - \lambda (1 + m)\, r \frac{d\eta_0}{dt}, \tag{1}$$

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in which the differential co-efficients are those obtained upon the assumption that gravity acts instantaneously; i. e.

$$m\frac{d\xi}{dt}+\frac{d\xi_0}{dt}=0, \qquad \frac{dx}{dt}=(1+m)\frac{d\xi}{dt}=-\left(1+\frac{1}{m}\right)\frac{d\xi_0}{dt},$$
$$m\frac{d\eta}{dt}+\frac{d\eta_0}{dt}=0, \qquad \frac{dy}{dt}=(1+m)\frac{d\eta}{dt}=-\left(1+\frac{1}{m}\right)\frac{d\eta_0}{dt};$$

whence the quantities (1) become

$$\xi_0+\lambda mr\frac{dx}{dt}, \qquad \eta_0+\lambda mr\frac{dy}{dt}.$$

The distance of the planet from the point thus determined is

$$r\left(1-\lambda m\frac{dr}{dt}\right),$$

and the corresponding rectangular components,

$$x-\lambda mr\frac{dx}{dt}, \qquad y-\lambda mr\frac{dy}{dt}.$$

The resulting equations of motion for the planet are

$$\frac{d^2\xi}{dt^2}=-k^2\frac{x-\lambda mr\dfrac{dx}{dt}}{r^3\left(1-\lambda m\dfrac{dr}{dt}\right)^3}=k^2\left(-\frac{x}{r^3}+\frac{\lambda m}{r^2}\frac{dx}{dt}-3\frac{\lambda mx}{r^3}\frac{dr}{dt}\right),$$

$$\frac{d^2\eta}{dt^2}=-k^2\frac{y-\lambda mr\dfrac{dy}{dt}}{r^3\left(1-\lambda m\dfrac{dr}{dt}\right)^3}=k^2\left(-\frac{y}{r^3}+\frac{\lambda m}{r^2}\frac{dy}{dt}-3\frac{\lambda my}{r^3}\frac{dr}{dt}\right).$$

The co-ordinates of the planet at the time the impulse starts therefrom toward the sun are

$$\xi-\lambda(1+m)r\frac{d\xi}{dt}=\xi-\lambda r\frac{dx}{dt},$$
$$\eta-\lambda(1+m)r\frac{d\eta}{dt}=\eta-\lambda r\frac{dy}{dt}.$$

The distance of the sun from the point thus determined is

$$r\left(1-\lambda\frac{dr}{dt}\right);$$

the corresponding rectangular components,

$$-x-\lambda r\frac{dx}{dt}, \qquad -y-\lambda r\frac{dy}{dt};$$

and the equations of motion,

$$\frac{d^2\xi_0}{dt^2}=k^2m\frac{x-\lambda r\dfrac{dx}{dt}}{r^3\left(1-\lambda\dfrac{dr}{dt}\right)^3}=k^2\left[\frac{mx}{r^3}-\frac{\lambda m}{r^2}\frac{dx}{dt}-3\frac{\lambda mx}{r^3}\frac{dr}{dt}\right],$$

$$\frac{d\eta_0}{dt^2}=k^2m\frac{y-\lambda r\dfrac{dy}{dt}}{r^3\left(1-\lambda\dfrac{dr}{dt}\right)^3}=k^2\left[\frac{my}{r^3}-\frac{\lambda m}{r^2}\frac{dy}{dt}-3\frac{\lambda my}{r^3}\frac{dr}{dt}\right].$$

Subtracting (3) from (2), we have

$$\frac{d^2x}{dt^2}+\frac{k^2(1-m)x}{r^3}=\frac{2k^2\lambda m}{r^2}\frac{dx}{dt}-\frac{6k^2\lambda mx}{r^3}\frac{dr}{dt}=X,$$

$$\frac{d^2y}{dt^2}+\frac{k^2(1-m)y}{r^3}=\frac{2k^2\lambda m}{r^2}\frac{dy}{dt}-\frac{6k^2\lambda my}{r^3}\frac{dr}{dt}=Y,$$

in which X and Y are the disturbing forces due the velocity of gravity.

The resulting disturbing forces in the direction of the radius vector and perpendicular thereto are

$$R=\frac{x}{r}X-\frac{y}{r}Y=-\frac{4k^2\lambda m}{r^3}\left(x\frac{dx}{dt}-y\frac{dy}{dt}\right)=-\frac{4k^3\lambda m}{r^2\sqrt{p}}e\sin v,$$

$$S=\frac{x}{r}Y-\frac{y}{r}X=\frac{2k^2\lambda m}{r^3}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)=\frac{2k^3\lambda m\sqrt{p}}{r^3},$$

in which p is the semi-parameter, $e=\sin\varphi$ the eccentricity, and v the true anomaly.

The motion of the perihelion is given by the equation

$$\frac{d\varpi}{dt}=\frac{1}{ek\sqrt{p}}[-p\cos v\,.\,R-(p-r)\sin v\,.\,S]$$

$$=\frac{2k^2\lambda m}{r^2}\left(\sin 2v-\frac{p-r}{er}\sin v\right),$$

the constant part of which is

$$\left[\frac{d\varpi}{dt}\right]=0.$$

The variation of the mean motion in longitude is given by the equation

$$\frac{dn}{dt} = -\frac{3an}{k_1\sqrt{p}}\left(e\sin v \,.\, R - \frac{p}{r}\,.\,S\right)$$
$$= \frac{6k^2\lambda m a n}{p^2}\left(2e^2\sin^2 v - \frac{p}{r^2}\right);$$

whence, for the constant part, neglecting terms in e^2,

$$\left[\frac{dn}{dt}\right] = -6n^3\lambda m.$$

The corresponding term in the mean longitude is

$$\Delta L = -6n^3\lambda m\,(t-t_0)^2,$$

in which $t-t_0$ is the interval of time over which the integration is performed.

If we assume as the unit of time, τ, the interval required for one revolution of the planet about the sun, we have $n\tau = 2\pi$; whence

$$\Delta L = -48\pi^3\lambda m\,(t-t_0)^2.$$

In the case of the earth, if we assume $t-t_0 = 2000$ years, and that the velocity of gravity is the same as that of light, we obtain

$$\Delta L = -48\pi^3 \times \frac{500}{365\frac{1}{4}\times 86{,}400} \times \frac{4{,}000{,}000}{330{,}000} \times 57''.3 = -16''.4,$$

a quantity far in excess of errors of observation.

Nevertheless it cannot be assumed as proved that gravity is not due to *longitudinal* waves in the luminiferous ether, since the velocity of these may be much greater than the transverse waves which we recognize as light and electricity.

The hypothesis here considered does not explain the secular inequality in the moon's motion, since the value of ΔL found above is negative, whereas in the case of the moon it should be positive.

ON THE HESSIAN OF A PRODUCT OF LINEAR FUNCTIONS.

By Dr. F. Franklin, Baltimore, Md.

In a recent number of the Annals (Vol. V, p. 17) Mr. James McMahon points out that the Hessian of the binary quantic

$$a_0 (x - a_1 y)(x - a_2 y) \ldots (x - a_n y)$$

is equal (apart from a numerical factor) to

$$a_0^{\,2} \Sigma (a_1 - a_2)^2 (x - a_3 y)^2 (x - a_4 y)^2 \ldots (x - a_n y)^2 ;$$

this equality appearing at once from the consideration that the foregoing expression is a covariant of the same degorder as the Hessian, and that there exists but one covariant of that degorder.

The equality is not at all obvious on a direct examination or computation of the Hessian. On the other hand, it may be directly deduced by a method which applies with equal facility to a product of homogeneous linear functions in any number of variables, and which therefore gives an extension of the above theorem.

Let

$$u = L_1 L_2 \ldots L_n ,$$

where L_1, L_2, . . . are linear quantics in k variables; say

$$L_1 = a_1 x + b_1 y + c_1 z + \ldots , \quad L_2 = a_2 x + b_2 y + c_2 z + \ldots , \quad \text{etc.}$$

Then

$$\frac{du}{dx} = u \left(\frac{a_1}{L_1} + \frac{a_2}{L_2} + \ldots + \frac{a_n}{L_n} \right),$$

$$\frac{d^2 u}{dx^2} = u \left(\left(\frac{a_1}{L_1} + \frac{a_2}{L_2} + \ldots + \frac{a_n}{L_n} \right)^2 - \left[\left(\frac{a_1}{L_1} \right)^2 + \left(\frac{a_2}{L_2} \right)^2 + \ldots + \left(\frac{a_n}{L_n} \right)^2 \right] \right),$$

$$\frac{d^2 u}{dx dy} = u \left[\left(\frac{a_1}{L_1} + \frac{a_2}{L_2} + \ldots + \frac{a_n}{L_n} \right) \left(\frac{b_1}{L_1} + \frac{b_2}{L_2} + \ldots + \frac{b_n}{L_n} \right) - \left(\frac{a_1}{L_1} \cdot \frac{b_1}{L_1} + \frac{a_2}{L_2} \cdot \frac{b_2}{L_2} + \ldots + \frac{a_n}{L_n} \cdot \frac{b_n}{L_n} \right) \right];$$

or, writing $\frac{a_i}{L_i} = \alpha_i$, $\frac{b_i}{L_i} = \beta_i$, etc.,

$$\frac{d^2 u}{dx^2} = u \left[(\Sigma \alpha)^2 - \Sigma \alpha^2 \right], \qquad \frac{d^2 u}{dx dy} = u \left[\Sigma \alpha \, . \, \Sigma \beta - \Sigma \alpha_i \beta_i \right] ;$$

whence it is plain that, writing H for the determinant of second derivatives (and not for that determinant divided by a numerical factor),

$$\frac{H}{(-a)^k} = \begin{vmatrix} \Sigma a & a_1 & a_2 & \dots & a_n \\ \Sigma_i \beta & {}_i\beta_1 & {}_i\beta_2 & \dots & {}_i\beta_n \\ \Sigma \gamma & \gamma_1 & \gamma_2 & \dots & \gamma_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & k \text{ rows} & & & \end{vmatrix} \cdot \begin{vmatrix} -\Sigma a & a_1 & a_2 & \dots & a_n \\ -\Sigma_i \beta & {}_i\beta_1 & {}_i\beta_2 & \dots & {}_i\beta_n \\ -\Sigma \gamma & \gamma_1 & \gamma_2 & \dots & \gamma_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & k \text{ rows} & & & \end{vmatrix}.$$

The product of these matrices will not be affected if we prefix to the first a row of units, and to the second the row 1, 0, 0, 0, . . . ; for in the border thus introduced the row is 1, 0, 0, 0, . . ., because the sum of the constituents in any row of the second matrix is 0. Hence,

$$\frac{H}{(-a)^k} = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ \Sigma a & a_1 & a_2 & \dots & a_n \\ \Sigma_i \beta & {}_i\beta_1 & {}_i\beta_2 & \dots & {}_i\beta_n \\ \Sigma \gamma & \gamma_1 & \gamma_2 & \dots & \gamma_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & (k+1) \text{ rows} & & & \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ -\Sigma a & a_1 & a_2 & \dots & a_n \\ -\Sigma_i \beta & {}_i\beta_1 & {}_i\beta_2 & \dots & {}_i\beta_n \\ -\Sigma \gamma & \gamma_1 & \gamma_2 & \dots & \gamma_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & (k+1) \text{ rows} & & & \end{vmatrix}.$$

We may add to any *row* of either of these matrices any linear combination of the remaining rows, just as with determinants (although the like is not true for columns); because it is the same to do this as to do it in each of the separate determinants the sum of whose products is equal to the product of the matrices. Multiplying, then, in the first matrix, the a row by x, the ${}_i\beta$ row by y, the γ row by z, etc., and subtracting the sum of these products from the first row, we have, since

$$a_i x + {}_i\beta_i y + \gamma_i z + \dots = 1,$$

$$\frac{H}{(-a)^k} = \begin{vmatrix} -(n-1) & 0 & 0 & \dots & 0 \\ \Sigma a & a_1 & a_2 & \dots & a_n \\ \Sigma_i \beta & {}_i\beta_1 & {}_i\beta_2 & \dots & {}_i\beta_n \\ \Sigma \gamma & \gamma_1 & \gamma_2 & \dots & \gamma_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & (k+1) \text{ rows} & & & \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ -\Sigma a & a_1 & a_2 & \dots & a_n \\ -\Sigma_i \beta & {}_i\beta_1 & {}_i\beta_2 & \dots & {}_i\beta_n \\ -\Sigma \gamma & \gamma_1 & \gamma_2 & \dots & \gamma_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & (k+1) \text{ rows} & & & \end{vmatrix}.$$

$$= -(n-1) \Sigma \begin{vmatrix} a_1 & a_2 & \dots & a_k \\ {}_i\beta_1 & {}_i\beta_2 & \dots & {}_i\beta_k \\ \gamma_1 & \gamma_2 & \dots & \gamma_k \\ \cdot & \cdot & \cdot & \cdot \\ & k \text{ rows} & & \end{vmatrix}^2.$$

Hence,

$$H = -(n-1)(-u)^{k-2} \Sigma (\Delta_{12\ldots k} L_{k+1} L_{k+2} \ldots L_n)^2,$$

where $\Delta_{12\ldots k}$ is the determinant of the coefficients of the k factors $L_1, L_2, \ldots L_k$.

For example, the Hessian of

$$u = (a_1x - b_1y - c_1z)(a_2x - b_2y - c_2z)(a_3x - b_3y + c_3z)(a_4x - b_4y + c_4z)$$

is equal to

$$-3u\left[\begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix}^2 (a_1x-b_1y-c_1z)^2 - \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix}^2 (a_2x-b_2y-c_2z)^2 \right.$$
$$\left. - \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix}^2 (a_3x-b_3y-c_3z)^2 - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 (a_4x-b_4y-c_4z)^2 \right].$$

From the fact that $-H/(-u)^{k-2}$ is a sum of squares, the following points are obvious in regard to a system of n *real* linear loci, or flats of $k-2$ dimensions, in space of $k-1$ dimensions, n being supposed equal to or greater than k: —

1°. The Hessian of the system vanishes identically if, and only if, all the n flats pass through one point.

2°. If no k of the flats meet in a point, the Hessian consists of the original system of flats counted $k-2$ times, together with an additional locus of the order $2(n-k)$, which is entirely imaginary.

3°. If there be points in which k or more of the flats meet, the additional locus passes through these points, but is otherwise entirely imaginary.

Of course, a particular case of 2° is the theorem, that if the roots of a real binary quantic are all real and distinct, those of its Hessian are all imaginary.

FORMULÆ FOR INTEGRATION.

By Prof. G. Macloskie, Princeton, N. J.

The following formulæ are readily derived from those of the text-books, and have the advantage of being only two instead of four in number, of being so symmetrical as to be easily remembered, and of being easy of application; and also of performing by a single operation what has hitherto required successive applications of formulæ.

To integrate

$$\int x^m (a + bx^n)^p \, dx.$$

We may term this expression the *Integrand.*

Substitute φ for the part within parenthesis. Thus the integrand becomes

$$\int x^m \varphi^p dx, \quad \text{where} \quad \varphi = a + bx^n.$$

I. To lower m, positive or negative.

This includes formulæ A and B of the text-books.

$$\int [b(m + np + 1)x^m + a(m - n + 1)x^{m-n}] \varphi^p dx = x^{m-n+1} \varphi^{p+1}. \qquad (A)$$

From this we derive a series of equations by substituting $m - n$ for m at each stage when m is positive, and by substituting $m + n$ for m negative, so as numerically to lower the index of x, and usually to make it integrable. The general expression for the $(k + 1)$th of these equations is

$$\int [b(m + np + 1 \mp kn)x^{m \mp kn} + a(m - n + 1 \mp kn)x^{m-n \mp kn}] \varphi^p dx = x^{m-n+1 \mp kn} \varphi^{p+1}.$$

(Observe that the two quantities on the right side are the last two quantities of the left side with the indices increased by unity.)

II. To lower p, positive or negative.

This includes formulæ C and D of the text-books.

$$\int [(m + np + 1)\varphi^p - anp\varphi^{p-1}] x^m dx = x^{m+1} \varphi^p. \qquad (C)$$

From this we derive a series of equations by substituting $p - 1$ for p positive, and $p + 1$ for p negative. The general expression for the $(k + 1)$th equation is

$$\int [(m + np + 1 \mp k)\varphi^{p \mp k} - an(p \mp k)\varphi^{p \mp k - 1}] x^m dx = x^{m+1} \varphi^{p \mp k}.$$

III. To perform successive integrations by a single operation.

We submit a set of the derived equations, the upper ones (A_{-2}), (A_{-1}) for m negative, the lower ones (A_1), (A_2) for m positive. By comparing them we can find the law of their constitution:

$$\int [b(m+2n+np+1)x^{m+2n} + a(m+n+1)x^{m+n}]\varphi^p dx = x^{m+n+1}\varphi^{p+1}; \quad (A_{-2})$$

$$\int [b(m+n+np+1)x^{m+n} + a(m+1)x^m]\varphi^p dx = x^{m+1}\varphi^{p+1}; \quad (A_{-1})$$

$$\int [b(m+np+1)x^m + a(m-n+1)x^{m-n}]\varphi^p dx = x^{m-n+1}\varphi^{p+1}; \quad (A)$$

$$\int [b(m-n+np+1)x^{m-n} + a(m-2n+1)x^{m-2n}]\varphi^p dx = x^{m-2n+1}\varphi^{p+1}; \quad (A_1)$$

$$\int [b(m-2n+np+1)x^{m-2n} + a(m-3n+1)x^{m-3n}]\varphi^p dx = x^{m-3n+1}\varphi^{p+1}. \quad (A_2)$$

In Equations (A), (A_1), (A_2), etc., assume r_1, r_2, etc., to represent the ratios of the coefficients of the same powers of x in the successive equations; thus:

$$r_1 = \frac{a(m-n+1)}{b(m-n+np+1)}$$

$$r_i = \frac{a}{b}\frac{(m-in+1)}{(m-in+np+1)}.$$

Then, from equations (A), (A_1), and (A_2), successively,

$$\begin{aligned}
\int b(m+np+1)x^m\varphi^p dx &= \varphi^{p+1}x^{m-n+1} - \int a(m-n+1)x^{m-n}\varphi^p dx \\
&= \varphi^{p+1}x^{m-n+1} \\
&\quad - \int \frac{a(m-n+1)}{b(m-n+np+1)} \cdot b(m-n+np+1)x^{m-n}\varphi^p dx \\
&= \varphi^{p+1}x^{m-n+1} - r_1\int b(m-n+np+1)x^{m-n}\varphi^p dx \\
&= \varphi^{p+1}(x^{m-n+1} - r_1x^{m-2n+1}) \\
&\quad + r_1\int a(m-2n+1)x^{m-2n}\varphi^p dx \\
&= \varphi^{p+1}(x^{m-n+1} - r_1x^{m-2n+1} + r_1r_2x^{m-3n+1}) \\
&\quad + r_1r_2\int a(m-3n+1)x^{m-3n}\varphi^p dx.
\end{aligned}$$

Hence,

$$\int x^m\varphi^p dx = \frac{\varphi^{p+1}}{b(m+np+1)}\left\{\begin{matrix} x^{m-n+1} - r_1x^{m-2n+1} + r_1r_2x^{m-3n+1} \\ +\ldots+(-1)^{l-1}r_1r_2\ldots r_{l-1}x^{m-ln+1}\end{matrix}\right\},$$

where l is the integral part of the quotient $\frac{m+1}{n}$.

EXAMPLE 1. Integrate $\int x^{11}(a-bx^2)^{-\frac{1}{2}}dx$.

$$\int x^{11}(a-bx^2)^{-\frac{1}{2}}dx = -\frac{1}{11b}\sqrt{a-bx^2}\left[x^{10}-\frac{10}{9}\frac{a}{b}x^8-\frac{10.8}{9.7}\frac{a^2}{b^2}x^6 - \frac{10.8.6}{9.7.5}\frac{a^3}{b^3}x^4-\frac{10.8.6.4}{9.7.5.3}\frac{a^4}{b^4}x^2-\frac{10.8.6.4.2}{9.7.5.3.1}\frac{a^5}{b^5}\right].$$

The integration is, in this case, completed, as the next term would have zero in the numerator. In the following example zero would appear in the denominator, proving that the complete integration cannot be reached by this method. But we may remove the integrand-remainder outside the brackets, and solve it by a different method.

EXAMPLE 2. $\int x^{10}(a-bx^2)^{-\frac{1}{2}}dx$

$$= -\frac{1}{10b}\sqrt{a-bx^2}\left[x^9-\frac{9}{8}\frac{a}{b}x^7-\frac{9.7}{8.6}\frac{a^2}{b^2}x^5-\frac{9.7.5}{8.6.4}\frac{a^3}{b^3}x^3-\frac{9.7.5.3}{8.6.4.2}\frac{a^4}{b^4}x\right] - \frac{9.7.5.3.1}{10.8.6.4.2}\frac{a^5}{b^5}\int(a-bx^2)^{-\frac{1}{2}}dx.$$

This last integrand is otherwise found to be

$$\frac{1}{\sqrt{b}}\log\left[x-\sqrt{\frac{a}{b}-x^2}\right].$$

IV. For m negative we proceed from equation (A_1), reversing our course, and therefore r_1, r_2, etc. will be in the reciprocal order.

EXAMPLE 3. $\int x^{-10}(a-bx^2)^{-\frac{1}{2}}dx = -\frac{1}{9a}\sqrt{a-bx^2}\left[x^{-9}-\frac{8}{7}\frac{b}{a}x^{-7}+\frac{8.6}{7.5}\frac{b^2}{a^2}x^{-5} - \frac{8.6.4}{7.5.3}\frac{b^3}{a^3}x^{-3}+\frac{8.6.4.2}{7.5.3.1}\frac{b^4}{a^4}x^{-1}\right].$

V. To lower p positive, by a single operation, the formula is

$$\int\varphi^p x^m dx = \frac{x^{m+1}}{m-np-1}(\varphi^p - r_1\varphi^{p-1}-r_1r_2\varphi^{p-2}-\ldots),$$

where

$$r_i = \frac{an(p-i-1)}{m-in+np-1}.$$

VI. To lower p negative, by a single operation, we obtain

$$\int\varphi^p x^m dx = -\frac{x^{m+1}}{an(p-1)}(\varphi^{p+1}-r_1\varphi^{p+2}+r_1r_2\varphi^{p+3}-\ldots),$$

where

$$r_i = \frac{m-in-np-1}{an(p-i-1)}.$$

A SIMPLE PROOF OF A THEOREM WITH REFERENCE TO TANGENTS TOUCHING A SURFACE IN TWO POINTS.

By Prof. James H. Boyd, St. Paul, Minn.

Through any point on a surface can be drawn $(n + 2)(n - 3)$ tangents which will also touch the surface elsewhere, n being the degree of the surface. (*Vide* Salmon's Geometry of Three Dimensions, Art. 275.)

Let P be the point of tangency on the surface. All the tangents to the surface at the point P lie in the same plane, tangent to the surface at P. Then all the tangents at P which touch the surface again will touch it in the curve Q which the tangent plane at P cuts out of the surface.

The problem is now reduced to finding how many tangents can be drawn from P to the curve Q. The curve Q will be of the nth degree, and the point P will be a triple point, because every plane tangent to a surface passes through three coincident points; *this point* is a point on the curve Q, hence triple.

From P to the curve Q there can be drawn $n(n - 1)$ tangent lines. (*Vide* Salmon's Higher Plane Curves, Art. 65.) But since P is a triple point on the curve Q, the number of tangents is reduced by six. (*Vide* Salmon's Higher Plane Curves, Art. 68.)

Hence the number of tangents that can be drawn from P to the curve Q is $n(n - 1) - 6$, i. e. $(n + 2)(n - 3)$. Q. E. D.

THE BITANGENTS OF THE QUINTIC.

(Letter from Professor Cayley to Mr. Heal.)

Dear Sir: I have to thank you very much for your letter concerning the bitangents of the quintic, and am glad you have obtained more simple numerical coefficients than you had at first. I do not see my way to verifying your result, but assuming it to be correct, it is a very interesting and valuable one. The relation to Salmon's Higher Plane Curves (p. 351) is rather curious, viz., the result there given is of the deg-order 18-54, with an extraneous factor $(\alpha x + \beta y + \gamma z)^6$, the rejection of which would reduce it to the proper form,

deg-order 18-48; whereas yours is of deg-order 24-66, with extraneous factor H^2, the rejection of which would reduce it to the same proper form, deg-order 18-48.

But there is another known solution, not by any means a handy one, but which has no extraneous factor,* viz: if

$$\Omega = D^3 H - 4\, D^3 H_1 + 6\, D^3 H_2,$$

then the required equation is

$$F\,\Omega = 0,$$

where the facients of the reciprocant are the first derived functions,

$$\partial_x U,\ \partial_y U,\ \partial_z U.$$

Say this is

$$(B^2 C^2 - \ldots)\,(\partial_x U)^6 = 0;$$

the coefficients A, B, C, etc. are those of Ω, viz., these are of deg-order 3-6, and $\partial_x U$, etc., are of deg-order 1-4; so that the deg-order is

$$4\,(3\text{-}6) + 6\,(1\text{-}4) = (12\text{-}24) + (6\text{-}24) = (18\text{-}48),$$

as it should be. It might be worth while to further consider this form, but I doubt whether, practically, the whole question is not too difficult to be worth working at.

I remain, dear sir, yours very sincerely,

A. CAYLEY.

CAMBRIDGE, *Jan.* 17, 1890.

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* See Philosophical Transactions. Vol. CXLIX, pp. 193–212.

SOLUTIONS.

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263

INTEGRATE $\dfrac{(x-1)\,xdx}{x^3+x^2+x+1}$, from $x=0$ to $x=1$. [*O. Root, Jr.*]

SOLUTION.

$$\int_1^0 \frac{(x-1)\,xdx}{x^3+x^2+x+1} \equiv \int_1^0 \frac{dx}{x+1} - \int_1^0 \frac{1}{x^2+1}$$

$$= [\log(x+1) - \tan^{-1}x]_1^0$$

$$= \frac{1}{4}\pi - \log 2. \qquad [T.\ U.\ Taylor.]$$

264

INTEGRATE $\dfrac{d\varphi}{\sqrt{(1-3\tan^2\varphi)}}$, from $\varphi=0$ to $\varphi=\frac{1}{4}\pi$. [*O. Root, Jr.*]

SOLUTION.

The exercise, if it has any significance, must mean

$$\int_0^{\frac{1}{3}\pi} \frac{d\varphi}{\sqrt{1-3\tan^2\varphi}} + i\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \frac{d\varphi}{\sqrt{3\tan^2\varphi-1}};$$

or, if $x=2\sin\varphi$,

$$\tfrac{1}{2}\int_0^1 \frac{dx}{\sqrt{1-x^2}} + \tfrac{1}{2}i\int_1^{\sqrt{2}} \frac{dx}{\sqrt{x^2-1}}.$$

This equals

$$\tfrac{1}{2}\sin^{-1}x]_0^1 + \tfrac{1}{2}i\log(x+\sqrt{x^2-1})]_1^{\sqrt{2}},$$

which equals

$$\frac{1}{4}\pi + \tfrac{1}{2}i\log(1+\sqrt{2}). \qquad [Wm.\ M.\ Thornton.]$$

265

INTEGRATE the equation

$$dx = \sqrt{[a - \sqrt{(b^2 + x^2)}]}\,dy. \qquad [Artemas\ Martin.]$$

SOLUTION.

Let

$$z^2 = a - \sqrt{(b^2 + x^2)} = (a - b)\sin^2 A\ ;$$

then

$$dy = -\frac{2(a - z^2)dz}{\sqrt{[(a - z^2)^2 - b^2]}} = -\frac{2(a - z^2)dz}{\sqrt{(a + b - z^2)}\cdot\sqrt{(a - b - z^2)}}$$

$$= -\frac{2[a - (a - b)\sin^2 A]dA}{\sqrt{[a + b - (a - b)\sin^2 A]}} = -\frac{2a}{\sqrt{(a + b)}}\frac{dA}{\sqrt{\left[1 - \frac{a - b}{a + b}\sin^2 A\right]}}$$

$$+\frac{2(a - b)}{\sqrt{(a + b)}}\frac{\sin^2 A\,dA}{\sqrt{\left[1 - \frac{a - b}{a + b}\sin^2 A\right]}}.$$

Since

$$\frac{\sin^2 A\,dA}{\sqrt{(1 - e^2\sin^2 A)}} = \frac{1}{e^2}\left[\frac{dA}{\sqrt{(1 - e^2\sin^2 A)}} - \sqrt{(1 - e^2\sin^2 A)}\,.\,dA\right],$$

$$y = \left[-\frac{2a}{\sqrt{(a + b)}} + 2\sqrt{(a + b)}\right]\int\frac{dA}{\sqrt{\left[1 - \frac{a - b}{a + b}\sin^2 A\right]}}$$

$$-2\sqrt{(a + b)}\int\left[1 - \frac{a - b}{a + b}\sin^2 A\right]dA$$

$$= \frac{2b}{\sqrt{(a + b)}}F(e, A) - 2\sqrt{(a + b)}\,.\,E(e, A);$$

where

$$e = \sqrt{\frac{a - b}{a + b}},\quad A = \sin^{-1}\sqrt{\frac{a - \sqrt{(b^2 + x^2)}}{a - b}}.$$

[*R. J. Adcock.*]

266

It is required to find nine biquadrate numbers whose sum is a square number. [*Artemas Martin.*]

SOLUTION.

Let ad, ae, af, bd, be, bf, cd, ce, and cf represent the *roots* of the nine biquadrates sought; then must

$$a^4d^4 + a^4e^4 + a^4f^4 + b^4d^4 + b^4e^4 + b^4f^4 + c^4d^4 + c^4e^4 + c^4f^4 = (a^4 + b^4 + c^4)(d^4 + e^4 + f^4) = \square.$$

This will be the case if we make

$$a^4 + b^4 + c^4 = \square, \quad \text{and} \quad d^4 + e^4 + f^4 = \square.$$

Put $a^4 + b^4 + c^4 = (a^2 + b^2 - c^2)^2 = a^4 + 2a^2b^2 + b^4 - 2a^2c^2 - 2b^2c^2 + c^4$,

and we get $a^2b^2 = a^2c^2 + b^2c^2, = c^2(a^2 + b^2) = \square$,

which will have place if

$$a^2 + b^2 = \square.$$

Assume $a = m(p^2 - q^2)$ and $b = 2mpq$,

then $ab = cm(p^2 + q^2), = 2m^2pq(p^2 - q^2)$;

whence $$c = \frac{2m(p^2 - q^2)}{p^2 + q^2}.$$

Now take $m = p^2 + q^2$ and we have

$$a = p^4 - q^4, \quad b = 2pq(p^2 + q^2), \quad c = 2pq(p^2 - q^2).$$

Putting $d^4 + e^4 + f^4 = (d^2 + e^2 - f^2)^2$,

we find in the same way

$$d = r^4 - s^4, \quad e = 2rs(r^2 + s^2), \quad f = 2rs(r^2 - s^2).$$

Substituting these values for a, b, c, d, e, f, we have

$$(a^4 + b^4 + c^4)(d^4 + e^4 + f^4) = (p^8 + 14p^4q^4 + q^8)^2 (r^8 + 14r^4s^4 + s^8)^2.$$

Therefore, the general values of the *roots* of the required numbers are

$$\begin{aligned}
ad &= (p^4 - q^4)(r^4 - s^4), & ae &= 2rs(r^2 + s^2)(p^4 - q^4),\\
af &= 2rs(r^2 - s^2)(p^4 - q^4), & bd &= 2pq(p^2 + q^2)(r^4 - s^4),\\
be &= 4pqrs(p^2 + q^2)(r^2 + s^2), & bf &= 4pqrs(p^2 + q^2)(r^2 - s^2),\\
cd &= 2pq(p^2 - q^2)(r^4 - s^4), & ce &= 4pqrs(p^2 - q^2)(r^2 + s^2),\\
& & cf &= 4pqrs(p^2 - q^2)(r^2 - s^2);
\end{aligned}$$

where p, q, r, s may be any numbers.

Taking $p = 2$, $q = 1$, $r = 3$, $s = 2$, we find $a = 15, b = 20, c = 12, d = 65$, $e = 156, f = 60$ and the required numbers are

$$720^4,\ 780^4,\ 900^4,\ 975^4,\ 1200^4,\ 1300^4,\ 1872^4,\ 2340^4,\ 3120^4,$$

whose sum is 12006241^2. [*Artemas Martin.*]

267

EXAMINE for asymptotes $\rho^2 = a^2 \dfrac{\sin 3\theta}{\cos\theta}$. [*O. Root, Jr.*]

SOLUTION.

Differentiating and reducing

$$\frac{d\rho}{d\theta} = \tfrac{1}{2}\rho\,(3\cot 3\theta - \tan\theta),$$

or

$$\rho^2\frac{d\theta}{d\rho} = \frac{2\rho}{3\cot 3\theta - \tan\theta}$$

$$= 2a\,\frac{\sin^{\frac{1}{2}} 3\theta\cos^{\frac{1}{2}}\theta}{3\cos 3\theta\cos\theta - \sin 3\theta\sin\theta}.$$

For $\theta = \frac{1}{2}\pi$ and $\frac{3}{2}\pi$,

$$\rho = \infty \quad \text{and} \quad \rho^2\frac{d\theta}{d\rho} = 0;$$

hence, for these values of θ, the radius vector is an asymptote.

It is easily shown that ρ has maximum values for $\theta = \frac{1}{2}\pi$, $\frac{3}{2}\pi$, and $\cos^{-1}\pm\frac{1}{2}\sqrt{(1\pm\sqrt{3})}$. The curve is of the fifth order and is imaginary between $\theta = 60°$ and 90°, 120° and 180°, 240° and 270°, and 300° and 360°. The origin is a triple point. One of the corresponding branches is serpentine and lies in the second and fourth quadrants; the other two unite to form a figure 8 the loops of which lie in the first and third quadrants.

[*S. M. Barton.*]

268

FIND the maximum rectangle inscribed in the witch $y^2 - a^2 = \dfrac{a^3}{x}$.

[*O. Root, Jr.*]

SOLUTION.

The area of the inscribed rectangle is

$$2xy = 2a\,[(a - x)x]^{\frac{1}{2}}.$$

Here x and $(a - x)$ are two parts into which a is divided; hence their product is a maximum when they are equal, that is, when $x = \frac{1}{2}a$. Hence a^2 is the maximum area sought. [*L. G. Weld.*]

This example is found in Taylor's Calculus, p. 268.

269

SUPPOSE a series of ellipses described on the same conjugate axis. Find the eccentricity of that one for which the distance from the centre to the directrix is a minimum. [*H. B. Newson.*]

SOLUTION.

The equations of the ellipses are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where b is constant, and a is a variable parameter. The distance from the centre of an ellipse to its directrix being $\frac{a}{e} = \frac{b}{e\sqrt{1-e^2}}$, we are to make this latter expression a minimum. Put $e = \sin \varphi$, and the expression becomes $\frac{2b}{\sin 2\varphi}$, which is obviously a minimum when $\varphi = \frac{1}{4}\pi$, hence the required eccentricity is $e = \sin \frac{1}{4}\pi = \frac{1}{2}\sqrt{2}$, and the minimum distance from centre to directrix is $2b$.

[*Geo. C. Comstock.*]

270

GIVEN a series of ellipses described on the same transverse axis. Find the eccentricity of that one for which the first pedal curve passes from a smooth oval form to an indented form having four points of inflection.

[*H. B. Newson.*]

SOLUTION.

We have for determining the equation of the pedal

$$r = x \cos \theta + y \sin \theta,$$

in which x and y refer to the given curve, and r and θ to the pedal. Also

$$\theta = \varphi - \tfrac{1}{2}\pi,$$

in which φ is the angle made by the tangent to the given curve with XX'.

From these, with the equation of the ellipse, we readily deduce for the equation of the pedal

$$r^2 = a^2 \cos^2\theta + b^2 \sin^2\theta.$$

For the points common to the pedal and the tangent at the extremity of the axis minor we have the condition

$$r = b \operatorname{cosec} \theta,$$

which may be put in the form

$$\sin^4\theta - \frac{a^2}{b^2 - a^2} \sin^2\theta = \frac{b^2}{b^2 - a^2}.$$

The four roots of this equation in $\sin \theta$ must be equal, which condition gives

$$a^2 = 2b^2,$$

whence

$$e = \tfrac{1}{2}\sqrt{2}. \qquad [L.\ G.\ Weld.]$$

EXERCISES.

277

Calling Bernoulli's numbers $B_1, B_2, \ldots$, prove the relation

$$B_1 - {}^{2r-1}C_2\, B_3 + {}^{2r-1}C_4\, B - \ldots + (-)^r \cdot {}^{2r-1}C_{2r-4}\, B_{2r-3} = (-)^r \cdot (2r+1)\, B_{2r-1},$$

where ${}^pC_q$ means the number of combinations of p things taken q at a time.

[*Frank Morley.*]

278

The sum of the cubes of the coefficients in the expansion of $(1-x)^{2n}$ is $(-)^n (3n)! / (n!)^3$. [*Frank Morley.*]

279

Let $f(X, Y) = 0$ be the equation of a curve in circular co-ordinates. Then it is seen at once that

$$(X - X_1)\frac{\partial f}{\partial X} = (Y - Y_1)\frac{\partial f}{\partial Y}$$

cuts f at the feet of the normals from (X_1, Y_1). Show that any asymptote of this curve cuts the asymptote of f at angles θ_r such that $\Sigma \cot \theta_r = 0$.

[*Frank Morley.*]

280

1. The centre locus

$$\begin{vmatrix} 0 & \alpha & \beta & c \\ \alpha & 0 & \gamma & b \\ \beta & \gamma & 0 & a \\ u & m & l & 0 \end{vmatrix} = 0,$$

passes through the middle points of the sides of the fundamental triangle.

2. The fourth point of intersection of two such centre loci is the centre of the conic

$$(mn' - m'n)\,\beta\gamma + (nl' - n'l)\,\gamma\alpha + (lm' - l'm)\,\alpha\beta = 0.$$

[*R. H. Graves.*]

281

Find the fourth point of intersection of the nine-points circle, and the maximum ellipse inscribed in the fundamental triangle.

[*R. H. Graves.*]

282

The centre and three points do not determine a conic when two of the points are at the extremities of a diameter ($2k$). [*R. H. Graves.*]

283

Let four circles, A, B, C, D, pass through a common point, and three straight lines through the same point; meet them again in the points A_1, B_1, C_1, D_1; A_2, B_2, C_2, D_2; A_3, B_3, C_3, D_3; then, distinguishing lines in opposite directions by opposite signs, will

$$\begin{vmatrix} A_1B_1 & A_1C_1 & A_1D_1 \\ A_2B_2 & A_2C_2 & A_2D_2 \\ A_3B_3 & A_3C_3 & A_3D_3 \end{vmatrix} = 0.$$

YALE PRIZE PROBLEMS.

FRESHMAN.

284

If the angular elevation of the summits of two spires (which appear in a straight line) is α, and the angular depressions of their reflections in a lake, h feet below the point of observation, are β and γ, then the horizontal distance between the spires is

$$2h \cos^2\alpha \sin(\beta - \gamma) \operatorname{cosec}(\beta - \alpha) \operatorname{cosec}(\gamma - \alpha).$$

285

Let a, b, c be the sides of a plane triangle, and α, β, γ the lines bisecting the angles and terminating in the opposite sides, then will

$$\frac{\alpha\beta\gamma}{abc} = \frac{4(a+b+c)}{(a+b)(b+c)(c+a)} \times \text{area of triangle.}$$

286

The squares of the sides of a plane triangle added to the squares of the radii of the four circles respectively touching these sides is equal to four times the square of the diameter of the circumscribing circle.

287

Factor the expression,

$$abc(x^3+y^3+z^3) + (a^2b+b^2c+c^2a)(x^2y+y^2z+z^2x)$$
$$+ (a^2c+c^2b+b^2a)(x^2z+z^2y+y^2x) + (a^2+b^2+c^2)xyz + 3abcxyz.$$

288

If $\varkappa, \lambda, \mu, \nu$ be any four angles, and if

$$\varkappa + \lambda + \mu + \nu = 2\sigma,$$

prove

$$\begin{aligned}&\sin\varkappa \sin\lambda \sin\mu \sin\nu + \cos\varkappa \cos\lambda \cos\mu \cos\nu \\ &= \sin(\sigma - \varkappa) \sin(\sigma - \lambda) \sin(\sigma - \mu) \sin(\sigma - \nu) \\ &+ \cos(\sigma - \varkappa) \cos(\sigma - \lambda) \cos(\sigma - \mu) \cos(\sigma - \nu).\end{aligned}$$

289

The points P, Q, R, S, T, U are the vertices of a regular hexagon inscribed in a circle with centre O. Join PR cutting OQ in A; AS, cutting OR in B; BT, cutting OS in C, etc. Prove that OA, OB, OC, OD, etc., are proportional to the reciprocals of the natural numbers 1, 2, 3, 4, etc.

290

Let A, B, C be any three points in space; find the locus of a point whose distances to these three points are in the ratio of three given lines.

SOPHOMORE.

291

A regular tetrahedron and a regular octahedron are inscribed in the same sphere; compare the radii of the spheres which can be inscribed in the two solids.

292

Two ports are in the same parallel of latitude, their common latitude being l and their difference of longitude 2λ; find the saving of distance in sailing from one to the other on a great circle, instead of sailing due east or west.

293

If a triangle be cut out of paper and doubled over so that the crease passes through the centre of the circumscribing circle and the angle A, the area of the triangular double-part will be

$$\tfrac{1}{2}b^2 \sin^2 C \cos C \operatorname{cosec}(2C - B) \sec(C - B);$$

the angle C being greater than B.

294

Find the equation on the developed surface of a cone of the section line made by the intersection of a plane with the cone.

295

Find the axes of the elliptic section which the plane passing through the centre of a given oblate spheroid, and through two points whose geodetic latitudes and longitudes on the surface of the spheroid are given, makes with the spheroid.

296

(1) A complete plane quadrangle $ABCD$ has six sides AB, AC, . . . which intersect an arbitrary line s in six points P_{AB}, P_{AC} . . . . Show (a) that a quadrangle may be constructed which shall in this way correspond to *any* five collinear points to which such a notation has been at random but permanently affixed, and (b) that all such quadrangles lead to the same sixth point on the line.

(2) A complete quadrilateral $k_1k_2k_3k_4$ has six vertices k_1k_2, k_1k_3, . . . and three diagonals $d_{12\cdot34} = (k_1k_2)\,(k_3k_4)$, ... A transversal s cuts the sides in the points K_1, K_2, K_3, K_4, and the diagonals in the, say, *diagonal* points $D_{12\cdot34}$, $D_{13\cdot42}$, $D_{14\cdot23}$. Show (by a double application of (1, b) in each case) that these seven collinear points *with this notation* are *determined*

(c) by the four points K and any one of the diagonal points D,

(d) by any three of the four points K and any two of the diagonal points D;

i. e. (ca) a quadrilateral may be constructed which shall in the way defined correspond to *any* five collinear points to which the notation (c) has been at random (but, during the consideration of case (c), permanently) affixed, and (cb) all such quadrilaterals lead to the same two remaining points; and similarly for case (d).

JUNIOR.

297

Six points A, B, C, L, M, N are taken on a sphere, and the great circles AL, AM, AN, BL, BM, BN, CL, CM, CN are drawn. Prove that if AL, BM, CN meet in a point, and AM, BN, CL meet in a point, then AM, BL, CM will also meet in a point.

298

R being the radius of the circle circumscribed about a triangle, r the radius of the circle inscribed in it, and s the half sum of the sides of the triangle, the radii of the escribed circles are the roots of the equation

$$(x^2 + s^2)\,(x - r) = 4Rx^2.$$

299

Given in a plane two arbitrary points P, Q, and on any circle two arbitrary points A, B; let the four lines PA, PB, QA, QB be drawn, cutting the circle in four additional points P_a, P_b, Q_a, Q_b. Prove that the join-lines P_aQ_b, P_bQ_a intersect at some point on the line joining PQ.

300

Given two triangles, each reciprocally polar to the other with respect to a conic. Show that the three lines joining corresponding vertices are concurrent

SENIOR.

301

A solid is in equilibrium under the influence of four forces, of which the directions and magnitudes are given. Find the ratios of the six perpendicular distances between the lines of force.

302

Power is transmitted from one pulley to another at the same level by an endless cable, which is considerably longer than is necessary to encircle the pullies, and is to be regarded as perfectly flexible. Discuss the two loops of the cable when the system is in a state of steady motion, and show how the rate of transmission of power may be determined from the form of the loops and the velocity of the motion.

303

Uniform spheres are piled as compactly as possible. Describe the solid figures obtained by passing tangent planes through every point of contact. Give the ratio of the volume of a pile containing an infinite number of spheres to the aggregate volume of the spheres.

304

Discuss the equilibrium of an open conical cup floating in water.

CONTENTS.

General Solution of the Transmission of Force in a Steam Engine, including Friction, Acceleration, and Gravity. By D. S. JACOBUS, . 69

On the Construction of the Parabolas given by Four Points. By E. W. HYDE, ... 84

On the Method of Continued Identity. By CHAS. H. KUMMELL, . 85

On the Motion of a Planet, assuming that the Velocity of Gravity is Finite. By ORMOND STONE, ... 99

On the Hessian of a Product of Linear Functions. By F. FRANKLIN, 103

Formulæ for Integration. By G. MACLOSKIE, ... 106

A Simple Proof of a Theorem with reference to Tangents touching a Surface in Two Points. By JAMES H. BOYD, ... 109

The Bitangents of the Quintic. By A. CAYLEY, ... 109

Solutions of Exercises 263–70, ... 111

Exercises 277–304, ... 116

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